A • GUIMIER

About mathematical foundation

of the

special relativity

Abstract :
We give an axiomatic way to present the special relativity .

Contents :

(1) Context

This text is a enhanced partial translation of :
https ://archive • org/details/matricesdelorentz/mode/2 up

(2) Basic principles :

(α) *We consider a point (or observer)* $\boldsymbol{O}$, *in an affine space* $\boldsymbol{E}$, *which models our physical spatial space, of direction* $\vec{E}$, *vector space in* **3** *- dimensional Euclidean isomorphic to* $\mathbb{R}^3$ • *We associate* $\boldsymbol{O}$ *with an orthogonal coordinate system* $\boldsymbol{R}(\boldsymbol{O}, \boldsymbol{x}, \boldsymbol{y}, z)$ *with a base* $\boldsymbol{B}\left(\boldsymbol{O}, \vec{i}, \vec{j}, \vec{k}\right)$ *with its natural Euclidean structure.*
We provide $\boldsymbol{O}$ *with a clock which measures time* $\boldsymbol{t}$.
We suppose that at each point, fixed with respect to $\boldsymbol{O}$, *of the coordinate system* $\boldsymbol{R}$ *is associated with a clock synchronized with that of* $\boldsymbol{O}$ *which measures the same time* $\boldsymbol{t}$.
Synchronizing Clocks allows to have a variable $\boldsymbol{t}$ *independent.*
We assume that $\boldsymbol{R}$ *is Galilean that is to say if a moving point , left to itself, on which acts no force, continues its trajectory in straight line, at one uniform speed* • *This hypothesis implies the existence of one only time, except for a change of origin or a change of unit.*
We can thus construct a vector space with **4** *dimensions and a frame* $\mathscr{R}\left(\boldsymbol{O}_{t=0}, \boldsymbol{t}, \boldsymbol{x}, \boldsymbol{y}, z\right)$, $\boldsymbol{O}_{t=0}$ *representing the point* $\boldsymbol{O}$ *at time* $\boldsymbol{t=0}$ *in this space, associated with a base* $\mathscr{B}\left(\boldsymbol{O}_{t=0}, \boldsymbol{c}\overline{\boldsymbol{\tau}}, \overline{\boldsymbol{i}}, \overline{\boldsymbol{j}}, \overline{\boldsymbol{k}}\right)$ *orthonormal for the bilinear symmetric form form ,* $\boldsymbol{\phi}(\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{y}, z) = \boldsymbol{c}^2\boldsymbol{t}^2 - \boldsymbol{x}^2 - \boldsymbol{y}^2 - z^2$ *(***c*** being the speed of light)) .*

(β) *We consider another point (or observer)* $\boldsymbol{O'}$ *having a uniform speed* $\vec{V}$ *relative* ***to*** *O and measured by* $\boldsymbol{O}$ *with which is also associated an orthorormal coordinate system* $\boldsymbol{R'}(\boldsymbol{O'}, \boldsymbol{x'}, \boldsymbol{y'}, \boldsymbol{z'})$ *associated with a base* $\boldsymbol{B'}\left(\boldsymbol{O'}, \vec{i'}, \vec{j'}, \vec{k'}\right)$ *with its natural Euclidean structure.We provide* $\boldsymbol{O}$ *with a clock which measures time* $\boldsymbol{t}$.
We suppose that at each point, fixed with respect to $\boldsymbol{O'}$, *of the coordinate system* $\boldsymbol{R'}$ *is associated with a clock synchronized with that of* $\boldsymbol{O'}$ *which measures the same time* t' .
Synchronizing Clocks allows to have a variable $\boldsymbol{t'}$ *independent.*
We assume that $\boldsymbol{R'}$ *is Galilean.*
Also we can thus construct a vector space with **4** *dimensions and a frame* $\mathscr{R}'\left(\boldsymbol{O'}_{t=0}, \boldsymbol{t'}, \boldsymbol{x'}, \boldsymbol{y'}, z'\right)$, $\boldsymbol{O'}_{t'=0}$ *representing the point* $\boldsymbol{O'}$ *at time* $\boldsymbol{t'=0}$ *in this space, associated with a base* $\mathscr{B}\left(\boldsymbol{O'}_{t'=0}, \boldsymbol{c}\overline{\boldsymbol{\tau'}}, \overline{\boldsymbol{i'}}, \overline{\boldsymbol{j'}}, \overline{\boldsymbol{k'}}\right)$ *orthonormal for the bilinear symmetric form form ,* $\boldsymbol{\phi'}(\boldsymbol{t'}, \boldsymbol{x'}, \boldsymbol{y'}, z') = \boldsymbol{c}^2\boldsymbol{t'}^2 - \boldsymbol{x'}^2 - \boldsymbol{y'}^2 - z'^2$ *(***c*** being the speed of light)) .*

We will assume that the **2** *observers pass through the same point of* $\boldsymbol{E}$ *during their journey*

and at this time ,
O and O' set their clock to 0 We will assume that the 2 observers pass through the same point of E during their journey and at this time, O and O' set their clock to 0 ($t = t' = 0$).
This nonessential hypothesis simplifies the calculations and we will talk about the bases $\mathscr{B}\left(O\ ,\ c\overline{\tau}\ ,\ \overline{i}\ ,\ \overline{j}\ ,\ \overline{k}\right)$
and $\mathscr{B}'\left(O\ ,\ c\overline{\tau'}\ ,\ \overline{i'} ,\ \overline{j'} ,\ \overline{k'}\right)$ by setting $O = O_{t=0} = O'_{t'=0}$.
Otherwise we can consider a third observer O'', having the same uniform speed .
$\vec{V}$ relative to O , but whose trajectory, a straight line parallel to that of O', intersects that of O.
The spatio − temporal units being defined by the physical laws which we will suppose to be the same in the 2 frames , we will choose the same units in the 2 frames.
(γ) We assume that the photons move in a straight line at speed c, independently of the considered reference frame. We also assume that c is the maximum possible speed.
*This implies that for a photon emitted from O at time $t = 0$, that is to say also from O' at time $t' = 0$, its coordinates in $\mathscr{B}$ and $\mathscr{B}'$ **will** check simultaneously :*
$c^2t^2 - x^2 - y^2 - z^2 = 0 \Leftrightarrow c^2t'^2 - x'^2 - y'^2 - z'^2 = 0$ *(conservation of the cone of` light).*

*(**3**) Relative velocity of 2 frames :*
In classical mechanics, if we consider 2 observers O and O' in uniform relative motion to the other we can write that $\overrightarrow{V_{O'\mid O}} = -\overrightarrow{V_{O\mid O'}}$ for these 2 observers :
the time is absolute as well as the distance $\left\| \overrightarrow{OO'} \right\|$.
In special relativity , the laws of physics are the same in the 2 frames in uniform relative motion to the other , that is to say that same objects placed under the same conditions will produce the same effects :
Measuring velocity of O' relative to O and measuring the velocity of O relative to O' will give the same result as long as these 2 velocities have the same norm .
As the 2 measured times t and t' are different likewise for spatial coordinates,
Now remembering that at this stage of the study , we only know that the transformation is linear and that the velocity of light is invariant, we will justify in an elementary way that the relative velocity of the 2 frames in uniform translation has the same norm, measured in one or the other frame and vectorially opposite.
Let be two spatial frames $\mathscr{R}$ and $\mathscr{R}'$ in uniform translation. Let assume that their origins O and O' coincide only once at during their relative movement at a point in the spatial space and at this point the clocks of the two spatial frames are set to 0 : $t = t' = 0$.
We recall that the time associated with any fixed point by relation to O in $\mathscr{R}$ is synchronized with O .
Similarly for the time associated with any fixed point by relation to O' in $\mathscr{R}'$ is synchronized with O' .
We can thus define the uniform velocity of a point $P(t)$ with respect to O in $\mathscr{R}$ by:

$$\left(\vec{\mathcal{V}}_{P\mid O}\right)_{\mathscr{R}} = \frac{\overrightarrow{OP(t_1)} - \overrightarrow{OP(t_0)}}{t_1 - t_0}.$$ *Similarly in $\mathscr{R}'$.*

Knowing that $\overrightarrow{OO'}(0) = \vec{0}$, we have in $\mathscr{R}$:

$$\left(\vec{\mathcal{V}}_{O'\mid O}\right)_{\mathscr{R}} = \frac{\overrightarrow{OO'(t)}}{t} = -\frac{\overrightarrow{O'O(t)}}{t} = -\left(\vec{\mathcal{V}}_{O\mid O'}\right)_{\mathscr{R}} \quad ,$$

Similarly in $\mathscr{R}'$ knowing that $\overrightarrow{O'O(0)} = \vec{0}$:

$$\left(\vec{v}_{O\,|\,O'}\right)_{\mathcal{R}'} = \frac{\overrightarrow{O'O(t')}}{t'} = -\frac{\overrightarrow{OO'(t')}}{t'} = -\left(\vec{v}_{O'\,|\,O}\right)_{\mathcal{R}'} \quad .$$

Is true that : $\left(\vec{v}_{O'\,|\,O}\right)_{\mathcal{R}} = -\left(\vec{v}_{O\,|\,O'}\right)_{\mathcal{R}'}$?

We already know that these **2** *velocities are parallel to* $\overrightarrow{OO'}$*, constant and in opposite directions.*

Let us evaluate their respective norm Let $\vec{v}$ *be the vvelocity of* **O '***relative to O in* $\mathcal{R}$*.*

How to measure $\|\vec{v}\|$ *? For this we are going to carry out a simple experiment measured from* **O** *and* **O'***. We will note* $\mathcal{R}_4$ *and* $\mathcal{R}_4'$ *the frames in the space in* **4** *dimensions, associated with* $\mathcal{R}$ *and* $\mathcal{R}'$ *and at their respective clock .*

As the transformation which makes it possible to pass from $\mathcal{R}_4$ *to* $\mathcal{R}_4'$ *is linear , it can be represented by a matrix* **M***,* $\mathcal{R}_4$ *and* $\mathcal{R}_4$ *being provided with adequate orthonormal bases* $\mathcal{B}$ *and* $\mathcal{B}'$ *.*

Experiment :

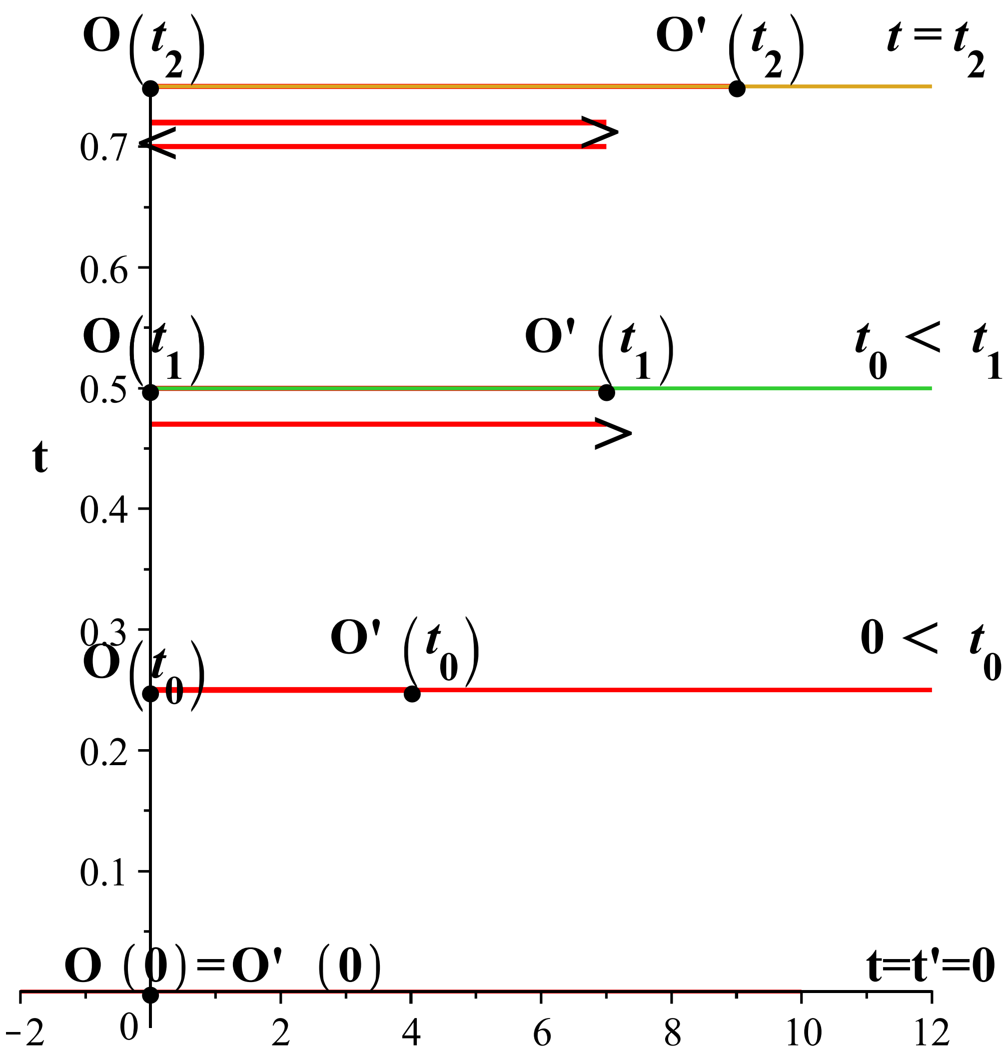
At time $t = t' = 0$, **O** *and* **O'** *coincide.*

At time $t = t_0 > 0$ *we send a light ray from* **O** *to* **O'** *which returns it to* **O** *at time* $t = t_1$,

and reaches **O** *at time* $t = t_2$.

We denote by $O'(t)$ *and* $O(t)$ *the position of* **O'** *and* **O** *at time* **t in** $\mathcal{R}$*.*

We assume that the axis $\overrightarrow{Ox}$ *is parallel to* $\overrightarrow{OO'}$ *:*



*As the movement is uniform rectilinear along the line defined **by O** and* $\vec{\mathcal{V}}$ *:*
we therefore have $\overline{OO'(t_0)} = \|\vec{\mathcal{V}}\|.t_0$, *similarly* $\overline{O'(t_0)O'(t_1)} = \|\vec{\mathcal{V}}\|.(t_1 - t_0)$.

The duration being the same on the outward and return journey $t_1 = t_0 + \frac{(t_2 - t_0)}{2} = \frac{(t_2 + t_0)}{2}$.

*If **c** is the speed of the light :*

$$(t_2 - t_0)c = 2\|\vec{\mathcal{V}}\|\ t_0 + 2\|\vec{\mathcal{V}}\|\ (t_1 - t_0)$$

therefore : $(t_2 - t_0)c = 2\|\vec{\mathcal{V}}\|\ t_1$ *and therefore :* $\|\vec{\mathcal{V}}\| = \frac{(t_2 - t_0)}{2\,t_1}c = \frac{(t_2 - t_0)}{t_2 + t_0}c$.

*From the observer's point of view on **O '**, he sees a ray starting from **O** at time* $t' = t_0'$,

and $\overline{O'O(t_0')} = \|\vec{\mathcal{V}'}\|.t_0'$,.

the ray will arrive at time t_1', *then it will depart towards **O** which it will touch at time* t_2'.

The ray will therefore have traveled the distance $c(t_2' - t_0')$.

In the direction $\overrightarrow{OO'}$ *the distance traveled is :* $\overline{O'O(t_0')} = \|\vec{\mathcal{V}'}\|.t_0'$ *and in the return direction :*

$$\|\vec{\mathcal{V}'}\|.t_0' + \|\vec{\mathcal{V}'}\|.(t_1' - t_0') + \|\vec{\mathcal{V}'}\|.(t_2' - t_1') = \|\vec{\mathcal{V}'}\|\ t_2' \text{ and therefore :}$$

$\left(t_2' - t_0'\right)c = \|\vec{\mathcal{V}}'\|\ t_1' + \|\vec{\mathcal{V}}'\|\ t_2'$ *and* $\|\vec{\mathcal{V}}'\| = \dfrac{\left(t_2' - t'_0\right)}{t_2' + t_0'}c$.

Let $M = \left(m_{i,j}\right)$ *be the transformation matrix from* $\mathcal{R}_4'$ *to* $\mathcal{R}_4$

O *has for coordinates* $(ct, 0, 0, 0)$ *in* $\mathcal{R}_4$,

O *will have for coordinates* $\left(m_{1,1}ct, m_{2,1}ct, m_{3,1}ct, m_{4,1}ct\right)$ *dans* $\mathcal{R}_4'$,

so $ct' = m_{1,1}ct$ *and therefore* $\|\vec{\mathcal{V}}\| = \dfrac{\left(t'_2 - t'_0\right)}{t'_2 + t'_0}.c = \dfrac{\left(t_2 - t_0\right)}{t_2 + t_0} \cdot c$.

has the same value in $\mathcal{R}'$ *and* $\mathcal{R}$.

The experiment can be seen by the **2** *observers* O *and* O' *as an experiment to measure the relative speed between* O *and* O' *we have :* $\left\|\vec{\mathcal{V}}_{O'|O}\right\|_{\mathcal{R}} = \left\|\vec{\mathcal{V}}_{O|O'}\right\|_{\mathcal{R}'}$.
We can therefore speak of the relative speed of **2** *frames in uniform translation* $\mathcal{R}$ *and* $\mathcal{R}'$ *with speed* $\vec{\mathcal{V}}$ *et* $-\vec{\mathcal{V}}$ *and with common module* $\|\vec{\mathcal{V}}\|$.

Note : *(See : N.M.J. Woodhouse .* " Special Relativity " *• Springer 2002)*
If we consider an observer on O *who observes a clock on* O' *which moves away from* O, *with a uniform speed and an observer on* O' *who observes an identical clock on* O *which moves away from* O' *we are in a completely symmetrical situation since we have the same speed in module in the two cases The physical laws being the same in the* **2** *Galilean frames in uniform translation the coefficient of expansion of the` durations will be the same in the* **2** *measurements made in each of the* **2** *frames.*
So if we denote $N = \left(n_{i,j}\right)$ *the inverse matrix of* M, *knowing that* $t' = m_{1,1}.t$ *et* $t = n_{1,1}.t'$,
by *what precedes we will have* $m_{1,1} = n_{1,1}$.

We will then denote γ *this common value .*

To go further we need :

Lemma : *If* $M = \left(m_{i,j}\right)$ *is symmetric matrix then :* $\left({}^tXMX = 0, \forall X \in \mathbb{R}^n\right) \Leftrightarrow M = 0$.

Proof : As M *is symmetric* M *can be diagonalized :* $M = {}^t\Omega D\Omega$ *where* Ω *is orthogonal and* D *diagonal then* ${}^tXMX = {}^tX\,{}^t\Omega D\Omega X = 0$.

If D *is such that* $d_{ij} = 0$ *for* $i \neq j$, *and* $Y = \Omega X$ *then* $\sum_{i=1}^{n} d_{ii} y_i^2 = 0$, $\forall Y \in \mathbb{R}^n$,

for $Y = e_i$, *with* $\left(e_i\right)_{i=1,\dots,n}$ *the canonical basis of* $\mathbb{R}^n$, *we have*

$d_{ii} \cdot 1 = 0 \Rightarrow d_{ii} = 0$ *• We can point out that the result is false*

if M *is not symmetric : take* $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$: ${}^tXMX = 0\ \ \forall X \in \mathbb{R}^n$ •

But we have the result $\left({}^tXMY, \forall X \in \mathbb{R}^n, \forall Y \in \mathbb{R}^n\right) \Leftrightarrow M = 0$,
since $e_i \cdot M \cdot e_j = m_{i,j}$ *• From this we can deduce again the lemma :*

If M *is symmetric and if we have* ${}^tXMX = 0, \forall X \in \mathbb{R}^n$ (1) *then* $\forall Y \in \mathbb{R}^n$ ${}^tYMY = 0$ (2) ,
and therefore ${}^t(X + Y)M(X + Y) = 0$ (3) ,
so ${}^tXMX + {}^tXMY + {}^tYMX + {}^tYMY = 0$ *and* ${}^tXMY + {}^tYMX = 0$.
M *being symmtreric and antisymmetric* $M = 0$.

<u>Corollary</u> : ${}^t(MX)G(MX)\ ,\ \forall X \in \mathbb{R}^n \Leftrightarrow {}^tMGM = G.$

Proof :

$${}^t(MX)G(MX) = {}^tX\,{}^tMGMX = {}^tXGX,\ \forall X \in \mathbb{R}^n \Leftrightarrow {}^tX\left({}^tMGM - G\right)X = 0,\ \forall X \in \mathbb{R}^n,$$

$$\Leftrightarrow {}^tMGM = G.$$

Lemma : (*See : N.M.J. Woodhouse* . " Special Relativity " *• Springer 2002*)

Let $A \in M_4(\mathbb{R})$ *be a symmetrc matrix and* $G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$

Let's assume $\forall X\ :\ {}^tXGX = 0 \Rightarrow {}^tXAX = 0$,

then $\exists\, \alpha \in \mathbb{R}$ *such as* $A = \alpha G$.

Proof **:**

If A *is symmetric we can write as :*

$A = \begin{bmatrix} \alpha & {}^ta \\ a & S \end{bmatrix}$ *where* $\alpha \in \mathbb{R},\ a \in M_{3,1}(\mathbb{R})$, $S \in M_3(\mathbb{R})$ S *symmetric* .

We compute for ${}^tX = [u, {}^tY],\ Y \in M_{3,1}(\mathbb{R})$, $r \in \mathbb{R}$ **:**

$${}^tXAX = [r, {}^tY]\begin{bmatrix} \alpha & {}^ta \\ a & S \end{bmatrix}\begin{bmatrix} r \\ Y \end{bmatrix} = \alpha r^2 + r\,{}^tYa + r\,{}^taY + {}^tYSY \text{ then :}$$

$${}^tXMX = \alpha t^2 + 2\,r\,{}^taY + {}^tYSY \ .(1)$$

We note ${}^tU = [u, v, w]$ *with* $u^2 + v^2 + w^2 = 1$ *et* ${}^tX = [1, U]$ (2)

In this case ${}^tXGX = 0$ *and then* ${}^tXAX = 0$,

and (1) *can be written with* $r = 1$, $Y = U$.

$\forall U$ *defined by* (2) **:** ${}^tXAX = \alpha + 2\,{}^taU + {}^tUSU = 0\ .(3)$

If U *verifies* (3) $-U$ *verifies* (2) *and* (3) *then:*

$$\alpha + 2\,{}^ta(-U) + {}^t(-U)S(-U) = 0 \ (4).$$

By adding (3) *and* (4) *it follows :*

$\forall U$ *defined by* (2) : $\alpha + {}^tUSU = 0 \Leftrightarrow {}^tU\left(\alpha Id_3 + S\right)U = 0.$

If ${}^tU\left(\alpha Id_3 + S\right)U = 0$ $\forall U$ *defined by* (2)

${}^tU\left(\alpha Id_3 + S\right)U = 0$ *is also true for* $\forall U \in \mathbb{R}^3$ **.**

According to the previous lemma : $S = -\alpha \cdot Id_3$.

By substracting (3) *et* (4) *we have :* $\forall U$ ${}^taU = 0 \Rightarrow a = 0$.

Then :

$$A = \alpha \cdot \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix} = \alpha G.$$

This result is a special case of a more general result :
Let be ϕ and ϕ' be two symmetric bilinear forms such the isotropic cone $\mathcal{C}(\phi) \neq \{0\}$
then $\exists \lambda \neq 0 \in \mathbb{R}\ \phi = \lambda\, \phi' \Leftrightarrow \mathcal{C}(\phi) = \mathcal{C}(\phi')$ **.**
(*cf* : *R.Goblot* ."Algébre linéaire "*Masson 1995* .)

Now we can prove :
Theorem : *Let* $M = (m_{i,j})$ *be the transformation matrix from* $\mathcal{R}_4{}'$ *to* $\mathcal{R}_4$
Let's assume $\forall X \ :\ {}^tXGX = 0 \Rightarrow {}^tX{}^tMGMX = 0 \ (1)$,
$$(M)_{1,1} = (M^{-1})_{1,1} \quad (2) \quad \text{then}$$
$${}^tMGM = G$$
Proof : (*cf* : *N • M • J • Woodhouse •* " Special Relativity " *•Springer 2002*)
From (1) *and the previous lemma , since* ${}^tMGM$ *is symmetric ,*
${}^tMGM = \alpha G$ *for some* $\alpha \in \mathbb{R}$, $\alpha \neq 0$ *since* M *is non singular.*
Hence $({}^tMGM)^{-1} = (\alpha G)^{-1} = \alpha^{-1} G$ *and*
$$M^{-1}G = \alpha^{-1} G\ {}^tM \Rightarrow M^{-1} = \alpha^{-1} G\ {}^tMG .$$
We have $(M)_{1,1} = ({}^tM)_{1,1} = (M^{-1})_{1,1} = (\alpha^{-1} G\ {}^tMG)_{1,1}$ *by* (2)
and
$$G\ {}^tMG = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} . \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$
Hence $(G\ {}^tMG)_{1,1} = m_{1,1}$ *and* $\alpha = 1$.

Note : (1) *is a consequence of the invariance of the light cone .*
For (2) *see the previous note .*

-

(*4*) *Lorentz matrix study :*
We start by giving the general properties of Lorentz matrices then
we give an estimation of each term of these matrices .
(*We note* ${}^tX$ *and* ${}^tM$ *the transpose of the column vector* X *and the matrix* M) .
Introduction :
Let $\Phi(X) = x_1^2 - \sum_{i=2}^{n} x_i^2$ *with* $({}^tX = (x_1, ..., x_n))$ *be the quadratic Lorentz*
form and let G *be the matrice* $\begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbb{R}^{n-1}} \end{bmatrix}$ *where* 0 *is the zero - column of* $\mathbb{R}^{n-1}$:
$\Phi(X) = {}^tXGX \ \ \forall X \in \mathbb{R}^n$.
We seek the matrices M *such*
$\Phi(MX) = \Phi(X) \Leftrightarrow {}^t(MX)G(MX) = {}^tX\,{}^tMGMX = {}^tXGX, \forall X \in \mathbb{R}^n$**.**

Lemma : *If* $M = (m_{i,j})$ *is symmetric then :* $({}^tXMX = 0, \forall X \in \mathbb{R}^n) \Leftrightarrow M = 0$.

Proof : As M *is symmetric* M *can be diagonalized :* $M = {}^t\Omega D\Omega$ *where* Ω *is orthogonal and* D *diagonal then* ${}^tXMX = {}^tX\,{}^t\Omega D\Omega X = 0$.

If D *is such that* $d_{ij} = 0$ *for* $i \neq j$, *and* $Y = \Omega X$ *then* $\sum_{i=1}^{n} d_{ii} y_i^2 = 0$, $\forall Y \in \mathbb{R}^n$,

for $Y = e_i$, *with* $(e_i)_{i=1,\ldots,n}$ *the canonical basis of* $\mathbb{R}^n$, *we have*

$d_{ii} \cdot 1 = 0 \Rightarrow d_{ii} = 0$. *We can point out that the result is false*

if M *is not symmetric : take* $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$: ${}^tXMX = 0 \;\; \forall X \in \mathbb{R}^n$.

But we have the result $({}^tXMY, \forall X \in \mathbb{R}^n, \forall Y \in \mathbb{R}^n) \Leftrightarrow M = 0$,
since $e_i \cdot M \cdot e_j = m_{i,j}$.*From this we can deduce again the lemma :*
If M *is symmetric and if we have* ${}^tXMX = 0, \forall X \in \mathbb{R}^n$ (1) *then* $\forall Y \in \mathbb{R}^n$ ${}^tYMY = 0$ (2) ,
and therefore ${}^t(X+Y)M(X+Y) = 0$ (3) ,
so ${}^tXMX + {}^tXMY + {}^tYMX + {}^tYMY = 0$ *and* ${}^tXMY + {}^tYMX = 0$.
M *being symmtreric and antisymmetric* $M = 0$.

<u>Corollary</u> : ${}^t(MX)G(MX)$, $\forall X \in \mathbb{R}^n \Leftrightarrow {}^tMGM = G$.
Proof :
$${}^t(MX)G(MX) = {}^tX\,{}^tMGMX = {}^tXGX, \forall X \in \mathbb{R}^n \Leftrightarrow {}^tX({}^tMGM - G)X = 0, \forall X \in \mathbb{R}^n,$$
$$\Leftrightarrow {}^tMGM = G.$$

<u>Definition</u> *:*

If M *is such* ${}^tMGM = G$, M *is called a Lorentz matrix* .

<u>Theorem 1 :</u>
The Lorentz matrices form a subgroup L *of* $GL_n(\mathbb{R})$, *group of the invertible matrices of dimension* n .

Proof : (1) *As* $Id_{\mathbb{R}^n} \in L$, $L \neq \varnothing$ *and* $\det({}^tMGM) = \det(G) = -1 \Rightarrow \det(M) \neq 0$,

$\Rightarrow \{Id_{\mathbb{R}^n}\} \subseteq L \subseteq GL_n(\mathbb{R})$.

(2) ${}^tMGM = G \Rightarrow G = G^{-1} = M^{-1}G({}^tM)^{-1} = M^{-1}G\,{}^t(M^{-1})$.
(3) *If* $M \in L$ *and* $N \in L$, *we have* ${}^t(MN)G(MN) = {}^tN({}^tMGM)N = G$.
From (1), (2) *and* (3) *we can infer the theorem .*
Corollary :

$${}^tMGM = G \Rightarrow M^{-1} = G\,{}^tM\,G = \begin{bmatrix} m_{1,1} & -m_{2,1} & -m_{3,1} & -m_{4,1} \\ -m_{1,2} & m_{2,2} & m_{3,2} & m_{4,2} \\ -m_{1,3} & m_{2,3} & m_{3,3} & m_{4,3} \\ -m_{1,4} & m_{2,4} & m_{3,4} & m_{4,4} \end{bmatrix} \text{ if } M = (m_{i,j}) .$$

A Lorentz matrix being invertible by (1) , M *has a unique polar decomposition :*
$M = SO$, S *symmetric and definite positive ,* O *orthogonal .*
We refer to a classical theorem : (theorem of decomposition)
Any invertible matrix A *can be written, in a unique way, as a product :* $A = SO = O_1S_1$,

where S, S_1 *are positive definite symmetric matrices* , O, O_1 *are orthogonal matrices,*

here : $S = \sqrt{{}^tAA}$, $S = \sqrt{A\,{}^tA}$.

(*Cf. F.R.Gantmacher : Theory of matrices. AMS Chelsea Publishing 1959*).

In the case of L *we can be more specific .*

For that we need some lemmas **.**

(*Cf. J - M. Souriau.*"Calcul Linéaire" • *PUF 1964 or J • Gabay 2000*)

Lemma 1 :

Let $X \in \mathbb{R}^n$ *be such that* ${}^tXX = 1$ • *Let* N *the matrix define by :*

$\mathcal{O}_{n-1}$ *being the null matrix of* $\mathbb{M}_{n-1}(\mathbb{R})$,

$$N = \begin{bmatrix} 0 & {}^tX \\ X & \mathcal{O}_{n-1} \end{bmatrix}$$ **.** *Then we have* $\forall \alpha \in \mathbb{R}$:

$$M = exp(\alpha N) = \begin{bmatrix} ch(\alpha) & sh(\alpha)\,{}^tX \\ sh(\alpha)X & \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha) - 1)X\,{}^tX\right) \end{bmatrix} \qquad (1)$$

is a Lorentz matrice of $\mathbb{M}_n(\mathbb{R})$ *such as* $\det(M) = 1$ *and its eingenvalues are strictly positive .*

In short M is a definite positive symmetric Lorentz matrix .

Proof : We have $N^2 = \begin{bmatrix} 1 & 0 \\ 0^{n-1} & X\,{}^tX \end{bmatrix}$ 0^{n-1} *being the zero - column of dimension* $n-1$,

since ${}^tXX = 1$ *and* $(X\,{}^tX)(X\,{}^tX) = X({}^tXX)\,{}^tX$ *we have* $N^3 = N$.

We can notice that for all square matrix A *and for all* $\alpha \in \mathbb{R}$,

$$exp(A) = \sum_{n=0}^{+\infty} \frac{A^n}{n!} \quad , \; sh(\alpha) = \sum_{n=0}^{+\infty} \frac{\alpha^{2n+1}}{(2n+1)!} \; , \; ch(\alpha) = \sum_{n=0}^{+\infty} \frac{\alpha^{2n}}{(2n)!} \; ,$$

therefore $exp(\alpha N) = Id_{\mathbb{R}^4} + \alpha N + \dfrac{\alpha^2 N^2}{2!} + \dfrac{\alpha^3 N}{3!} + \dfrac{\alpha^4 N^2}{4!} + \ldots\ldots + $

$$+ \frac{\alpha^{2p-1} N}{(2p-1)!} + \frac{\alpha^{2p} N^2}{(2p)!} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= Id_{\mathbb{R}^n} + sh(\alpha)N + (ch(\alpha) - 1)N^2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ | & \diagdown & | \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & sh(\alpha)\,{}^tX \\ sh(\alpha)X & \mathcal{O}_{n-1} \end{bmatrix} + \begin{bmatrix} (ch(\alpha) - 1) & 0 \\ 0^{n-1} & (ch(\alpha) - 1)\,X\,{}^tX \end{bmatrix}$$ *which shows* (1) .

Now let's show that M *is of Lorentz that is to say* ${}^tMGM = G$ **.**

We can remark that M *is symmetric and with* $Id = Id_{\mathbb{R}^{n-1}}$

$$\begin{bmatrix} ch(\alpha) & sh(\alpha)\,{}^tX \\ sh(\alpha)X & \left(Id + (ch(\alpha) - 1)X\,{}^tX\right) \end{bmatrix} \begin{bmatrix} 1 & {}^t0^{n-1} \\ 0^{n-1} & -Id \end{bmatrix} = \begin{bmatrix} ch(\alpha) & -\,sh(\alpha)\,{}^tX \\ sh(\alpha)X & -\left(Id + (ch(\alpha) - 1)X\,{}^tX\right) \end{bmatrix}$$

and

$$\begin{bmatrix} ch(\alpha) & -sh(\alpha)\,{}^tX \\ sh(\alpha)X & -\left(Id+(ch(\alpha)-1)X{}^tX\right) \end{bmatrix}\begin{bmatrix} ch(\alpha) & sh(\alpha)\,{}^tX \\ (sh(\alpha))X & \left(Id+(ch(\alpha)-1)X{}^tX\right) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Let us estimate **a, b, c, d** .

$a = ch^2(\alpha) - sh^2(\alpha)\,{}^tXX = 1$, *since* ${}^tXX = 1$ **.**

$$b = ch(\alpha)\ sh(\alpha)\,{}^tX - sh(\alpha)\,{}^tX\left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right)$$

$$= ch(\alpha)\ sh(\alpha)\,{}^tX - sh(\alpha)\,{}^tX - sh(\alpha)(ch(\alpha)-1)\,{}^tXX{}^tX$$

$$= ch(\alpha)\ sh(\alpha)\,{}^tX - sh(\alpha)\,{}^tX - sh(\alpha)ch(\alpha)\,{}^tX + sh(\alpha){}^tX = 0,$$

$$c = ch(\alpha)\ sh(\alpha)X - \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right)sh(\alpha)X$$

$$= ch(\alpha)\ sh(\alpha)X - sh(\alpha)X - sh(\alpha)(ch(\alpha)-1)X = 0,$$

$$d = sh^2(\alpha)X{}^tX - \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right)^2 \quad but:$$

$$\left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right)\left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right)$$

$$= Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX + (ch(\alpha)-1)X{}^tX + (ch(\alpha)-1)^2X{}^tXX{}^tX,$$

since ${}^tXX = 1$ *then* $X({}^tXX){}^tX = X{}^tX$ (*association of matrix product*) ,

et $(ch(\alpha)-1)^2 = ch^2(\alpha) - 2ch(\alpha) + 1$

$$= Id_{\mathbb{R}^{n-1}} + \left(2(ch(\alpha)-1) + ch^2(\alpha) - 2ch(\alpha) + 1\right)X{}^tX$$

$$= Id_{\mathbb{R}^{n-1}} + \left(ch^2(\alpha) - 1\right)X{}^tX = Id_{\mathbb{R}^{n-1}} + sh^2(\alpha)X{}^tX$$

finally :

$$sh^2(\alpha)X{}^tX - \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right)^2 = -Id_{\mathbb{R}^{n-1}}.$$

Therefore $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbb{R}^{n-1}} \end{bmatrix}$

and **M** *is of* **Lorentz** .

Another proof possible is :

$$exp(\alpha N)G = \begin{bmatrix} ch(\alpha) & sh(\alpha)\,{}^tX \\ sh(\alpha)X & \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right) \end{bmatrix}\begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbb{R}^{n-1}} \end{bmatrix}$$

$$= \begin{bmatrix} ch(\alpha) & -sh(\alpha)\,{}^tX \\ sh(\alpha)X & -\left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right) \end{bmatrix},$$

$$Gexp(-\alpha N) = \begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbb{R}^{n-1}} \end{bmatrix}\begin{bmatrix} ch(\alpha) & -sh(\alpha)\,{}^tX \\ -sh(\alpha)X & \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right) \end{bmatrix}$$

$$= \begin{bmatrix} ch(\alpha) & -sh(\alpha)\,{}^tX \\ sh(\alpha)X & -\left(Id_{\mathbb{R}n-1} + (ch(\alpha)-1)X{}^tX\right) \end{bmatrix}$$

therefore $\boldsymbol{exp(\alpha N)G = Gexp(-\alpha N)} \Rightarrow \boldsymbol{exp(\alpha N)Gexp(\alpha N) = G}$

as $\boldsymbol{M = exp(\alpha N)}$ *est symétrique* ${}^t\boldsymbol{MGM = G}$ *et* $\boldsymbol{M}$ *est de* **Lorentz.**

For a third demonstration see : J-M. Souriau."Calcul Linéaire " *•PUF 1964 or J •Gabay 2000* **.**

Let us now show that $\boldsymbol{det(exp(A)) = e^{Tr(A)}}$ *for* $\boldsymbol{A \in M_n(\mathbb{R})}$ *:*

Let us consider $\boldsymbol{A}$ *as an element of* $\boldsymbol{M_n(\mathbb{C})}$ *.* $\boldsymbol{A}$ *is then trigonalisable*

and can be written : $\boldsymbol{A = P^{-1}BP}$ *, with* $\boldsymbol{B}$ *upper triangular whose diagonal*

is composed of the eigenvalues $\boldsymbol{\lambda_i}$ *of* $\boldsymbol{A}$*.*

By relying on the definition of the exponential of a matrix we can write

$\boldsymbol{exp(A) = P^{-1}exp(B)P}$**.**

By expanding $\boldsymbol{exp(B)}$ *into series and by noticing that* $\boldsymbol{B^k}$ *is upper triangular* $\boldsymbol{\forall k \geq 1}$**.**

Then $\boldsymbol{exp(B)}$ *has for diagonal the* $\boldsymbol{e^{\lambda_i}} > 0$*. Therefore*

$$\boldsymbol{det(exp(A)) = det(exp(B)) = \prod_i e^{\lambda_i} = e^{\sum_i \lambda_i} = e^{Tr(A)}}\ .$$

Let's go back to $\boldsymbol{exp(\alpha N)}$ *:*

As $\boldsymbol{\alpha N}$ *is a zero − trace matrix :* $\boldsymbol{det(M) = det(exp(\alpha N)) = 1}$ *and the eingenvalues of*

$\boldsymbol{M = exp(\alpha N)}$ *are strictly positive .*

Lemma 2 :

If M *is a matrix of Lorentz whose first column* K_1 *is of the form :* $K_1 = {}^t(\alpha, 0, ..., 0)$ *then* M *is of the form :*

$$M = \begin{bmatrix} \varepsilon & {}^tQ \\ Q & \Omega \end{bmatrix} \qquad \varepsilon = \pm 1 \,,\, {}^tQ = (0, 0, 0),\ {}^t\Omega\Omega = Id_{\mathbb{R}^{n-1}} \quad (1).$$

Conversely any matrix of this form is of Lorentz.

Proof :

Let $M = \begin{bmatrix} \alpha & {}^tL \\ 0 & C \end{bmatrix}$ *with* $C \in \mathbb{M}_3(\mathbb{R})$, ${}^tL = (l_1, l_2, l_3)$.

As $({}^tMG)M = G$ *we have :*

$$G = \begin{bmatrix} \alpha & 0 \\ L & -C \end{bmatrix} \begin{bmatrix} \alpha & {}^tL \\ 0 & C \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^2 & \alpha\,{}^tL \\ \alpha L & L\,{}^tL - {}^tCC \end{bmatrix}$$

by identification $\alpha = \pm 1$, $L = 0$, ${}^tCC = Id_{\mathbb{R}^{n-1}}$.

Conversely : $\begin{bmatrix} \varepsilon & {}^tQ \\ Q & {}^t\Omega \end{bmatrix} G \begin{bmatrix} \varepsilon & {}^tQ \\ Q & \Omega \end{bmatrix} = G$.

Theorem :

Every matrix M *of Lorentz can be put in the form :*

$$M = exp(\alpha N) \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix} \quad \text{with } \alpha \in \mathbb{R},\ \varepsilon = \pm 1 ,$$

$N = \begin{bmatrix} 0 & {}^tX \\ X & O \end{bmatrix}$, *with* $X \in \mathbb{R}^{n-1}$ *such as* : ${}^tXX = 1$, ${}^t\Omega\Omega = Id_{\mathbb{R}^{n-1}}$.

This polar decomposition is unique .

Proof :

Let M *be a matrix of Lorentz* $M = (m_{i,j})$ *and* M_1 *the first column of* M.

If $M_1 = (m_{i,1})$ *then* :

$${}^tM_1GM_1 = m_{1,1}^{\ 2} - m_{1,2}^{\ 2} - ... - m_{1,n}^{\ 2} = G_{1,1} = 1 .$$

We write ${}^tM_1 = (\beta, Y)$ *with* $\beta = m_{1,1}$ *and* ${}^tY = (m_{1,2}, ..., m_{1,n})$.

Therefore $\beta^2 - {}^tYY = 1$ *and* $|\beta| \geq 1$ • *We can write* $\beta = \varepsilon \cdot ch(\alpha)$ *with* $\varepsilon = \pm 1$ *and* $\alpha \in \mathbb{R}$.*Moreover* ${}^tYY = \beta^2 - 1 = sh^2(\alpha)$.

If $\alpha \neq 0$ *put* $X = \dfrac{Y}{\varepsilon \cdot sh(\alpha)}$, *we have* ${}^tXX = 1$.

If $\alpha = 0$ *on a* ${}^tM_1 = (\varepsilon, 0, 0, 0)$.

In all cases ${}^tM_1 = \varepsilon(ch(\alpha), sh(\alpha)X)$ *avec* ${}^tXX = 1$.

M_1 *is the first column of* $\varepsilon \cdot exp(\alpha N)$ *with* $N = \begin{bmatrix} 0 & {}^tX \\ X & O \end{bmatrix}$.

$exp(-\alpha N) \cdot M$ *is a matrix of Lorentz since* M *and* $exp(-\alpha N)$ *are of Lorentz* (**lemma 1**).

Let's estimate the first column K_1 *of this product :*

$$K_1 = \begin{bmatrix} ch(\alpha) & (sh(-\alpha))\,{}^tX \\ (sh(-\alpha))X & Id_{\mathbb{R}^3} + (ch(\alpha) - 1)X{}^tX \end{bmatrix} \cdot M_1$$

${}^tXX = 1$, $ch^2(\alpha) - sh^2(\alpha) = 1$ *and*

$\left(Id_{\mathbb{R}^3} + (ch(\alpha) - 1)X{}^tX\right) \cdot X = ch(\alpha) \cdot X$ *since* $(X{}^tX)X = X({}^tXX)$

We have : ${}^tK_1 = (\varepsilon, 0, .., 0)$ *• The lemma 2 implies :*

$$M = exp(\alpha N) \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix}.$$

αN *being a symmetric matrix , it is a diagonalizable matrix :* $\alpha N = {}^tPDP$,
with D *a diagonal matrix and* P *an orthogonal matrix :*

therefore $exp(\alpha N) = exp({}^tPDP) = {}^tPexp(D)P$ *and*
$exp(\alpha N)$ *is a definite positive matrix .*

Moreover $\begin{bmatrix} \varepsilon & 0 \\ 0 & {}^t\Omega \end{bmatrix}\begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix} = Id_{\mathbb{R}^n}$.

We have decompozed M *into an product of a definite positive symmetric matrix and an orthogonal one ,by the theorem of decomposition , there is uniqueness .*

***Note** :*

(1) *If* $n = 2$ *then* $\Omega = \pm 1$ *and if* $\Omega \cdot \varepsilon = 1$ *then* M *is symmetric .*

If $n = 3$ *then* $\Omega = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$ *or* $\Omega = \begin{bmatrix} cos(\varphi) & sin(\varphi) \\ sin(\varphi) & -cos(\varphi) \end{bmatrix}$ *with* $\theta \in \mathbb{R}$, $\varphi \in \mathbb{R}$.

If $n = 4$ *then* Ω *is a spatial rotation if* $det(\Omega) = +1$.

If not Ω *is a reflection, or a combination of a rotation and a reflection .*

(2) *Let's consider* ${}^tW' = (ct', 0, 0, 0)$ *with* c *a constant ,* t' *a real variable* > 0 *and let* $W = MW'$.

Let 's assume that ${}^tW$ *can be written in the form :* ${}^tW = (ct, ct\beta_1, ct\beta_2, ct\beta_3)$
with $\beta_1, \beta_2, \beta_3$ *real constants ,* t *a real variable.*

If $M = \begin{bmatrix} \gamma & {}^{t}(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C \end{bmatrix} \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix}$ then $t = \varepsilon \cdot \gamma \cdot t'$ and $\varepsilon = \dfrac{t \cdot t'}{|t \cdot t'|}$.

Now, we are going to explicit M in the case of the special relativity :

Lemma 3 :

Let M be a matrix of Lorentz with $n = 4$, $\varepsilon = 1$ and let's consider ${}^{t}W' = (ct', 0, 0, 0)$ with c a constant , $t' > 0$ a real variable and let $W = MW'$.
Let 's assume that ${}^{t}W$ can be written in the form :
${}^{t}W = (ct, ct\beta_1, ct\beta_2, ct\beta_3)$ with $\beta_1, \beta_2, \beta_3$ real constants , t a real variable.

Then if $M = exp(\alpha N) \cdot \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}$, with $N = \begin{bmatrix} 0 & {}^{t}X \\ X & \mathcal{O}_{n-1} \end{bmatrix}$ and $\mathcal{O}_{n-1}$ the null matrix

then

if we write $\vec{\beta} = {}^{t}(\beta_1, \beta_2, \beta_3)$ and $\vec{X} = {}^{t}(X_1, X_2, X_3)$,

we have $exp(\alpha N) = \begin{bmatrix} \gamma & {}^{t}(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C \end{bmatrix}$ and then $M = \begin{bmatrix} \gamma & {}^{t}(\gamma\vec{\beta})\Omega \\ \gamma\vec{\beta} & C\Omega \end{bmatrix}$,

$$\text{with} \qquad C = \begin{bmatrix} 1 + \dfrac{\gamma^2}{(1+\gamma)}\beta_1^2 & \dfrac{\gamma^2}{(1+\gamma)}\beta_1\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\beta_1\beta_3 \\ \dfrac{\gamma^2}{(1+\gamma)}\beta_2\beta_1 & 1 + \dfrac{\gamma^2}{(1+\gamma)}\beta_2^2 & \dfrac{\gamma^2}{(1+\gamma)}\beta_2\beta_3 \\ \dfrac{\gamma^2}{(1+\gamma)}\beta_3\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\beta_3\beta_2 & 1 + \dfrac{\gamma^2}{(1+\gamma)}\beta_3^2 \end{bmatrix} ,$$

$\vec{\beta} = th(\alpha)\vec{X}$ and $\gamma = ch(\alpha)$.

Proof :

First we can point out that $\begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} W' = W'$ donc $W = exp(\alpha N) W'$.

We have to solve :

$$\begin{bmatrix} ch(\alpha) & sh(\alpha)\,{}^{t}X \\ sh(\alpha)X & \left(Id_{\mathbb{R}^3} + (ch(\alpha) - 1)X\,{}^{t}X\right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} t' = \begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} t \ .$$

This implies $t \begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = t' \begin{bmatrix} ch(\alpha) \\ sh(\alpha)X_1 \\ sh(\alpha)X_2 \\ sh(\alpha)X_3 \end{bmatrix}$ *therefore* $t = ch(\alpha)t'$ *and*

$ch(\alpha) \begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} ch(\alpha) \\ sh(\alpha)X_1 \\ sh(\alpha)X_2 \\ sh(\alpha)X_3 \end{bmatrix}$ *for* $t' \neq 0 \Rightarrow ch(\alpha)\vec{\beta} = sh(\alpha)\vec{X}$

therefore $\vec{\beta} = th(\alpha)\vec{X} \Rightarrow \vec{\beta}^2 = th^2(\alpha)$ *since* $\vec{X}^2 = 1$.

We put $\beta = \sqrt{\vec{\beta}^2}$ *and* $\gamma = \dfrac{1}{\sqrt{1 - \vec{\beta}^2}}$.

As $1 - th^2(\alpha) = \dfrac{1}{ch^2(\alpha)} \Rightarrow ch^2(\alpha) = \dfrac{1}{1 - th^2(\alpha)} = \dfrac{1}{1 - \vec{\beta}^2} = \gamma^2$,

As $ch(\alpha) \geq 1$ *we have* $\gamma = ch(\alpha)$;

As $sh^2(\alpha) = ch^2(\alpha) - 1 = \gamma^2 - 1 = \dfrac{1}{1 - \vec{\beta}^2} - 1 = \dfrac{\beta^2}{1 - \beta^2} = \gamma^2\beta^2$.

Let's sum up $\gamma = ch(\alpha)$, $\gamma^2\beta^2 = sh^2(\alpha)$, $\beta^2 = th^2(\alpha)$.
On the other hand :

$$\gamma^2\beta^2 = \gamma^2 - 1 = (\gamma + 1)(\gamma - 1) \Rightarrow \frac{\gamma^2\beta^2}{(1+\gamma)} = (\gamma - 1) = ch(\alpha) - 1$$

and $X_iX_j = \dfrac{(\beta_i\beta_j)}{th^2(\alpha)} = \dfrac{(\beta_i\beta_j)}{\beta^2}$ *therefore :*

$(ch(\alpha) - 1)X_iX_j = \dfrac{\gamma^2\beta^2}{(1+\gamma)}\dfrac{(\beta_i\beta_j)}{\beta^2} = \dfrac{\gamma^2}{(1+\gamma)}\beta_i\beta_j$, *that implies:*

$$Id_{\mathbb{R}^3} + (ch(\alpha) - 1)X^tX = \begin{bmatrix} 1 + \frac{\gamma^2}{(1+\gamma)}\beta^2_1 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_1 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta^2_2 & \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_1 & \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_2 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta^2_3 \end{bmatrix}$$

$$= Id_{\mathbb{R}^3} + \frac{\gamma^2 \vec{\beta}\, {}^t\overleftrightarrow{\beta}}{(1+\gamma)} = C$$

As $sh(\alpha)X_i = \dfrac{sh(\alpha)\beta_i}{th(\alpha)} = ch(\alpha)\beta_i = \gamma\beta_i$ finally we have

$$exp(\alpha N) = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C \end{bmatrix}.$$

Then $M = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C \end{bmatrix} . \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta})\Omega \\ \gamma\vec{\beta} & C\Omega \end{bmatrix}.$

Corollary : As $M^{-1} = G\,{}^tM\,G = \begin{bmatrix} m_{1,1} & -m_{2,1} & -m_{3,1} & -m_{4,1} \\ -m_{1,2} & m_{2,2} & m_{3,2} & m_{4,2} \\ -m_{1,3} & m_{2,3} & m_{3,3} & m_{4,3} \\ -m_{1,4} & m_{2,4} & m_{3,4} & m_{4,4} \end{bmatrix}$ if $M = (m_{i,j})$,

$M^{-1} = \begin{bmatrix} \gamma & -{}^t(\gamma\vec{\beta}) \\ -\gamma\,{}^t\Omega\vec{\beta} & {}^t\Omega C \end{bmatrix}$, M^{-1} is of Lorentz $\Rightarrow M^{-1} = \begin{bmatrix} \gamma & -{}^t(\gamma\vec{\beta})\Omega' \\ -\gamma\vec{\beta} & C\Omega' \end{bmatrix}$

then ${}^t\Omega\vec{\beta} = \vec{\beta} \Rightarrow {}^t\overleftrightarrow{\beta}\Omega = {}^t\overleftrightarrow{\beta}$ and $M = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C\Omega \end{bmatrix}.$

Note : If $C\Omega = [u_1, u_2, u_3]$ then $u^2_1 + u^2_2 + u^2_3 = 3 + \gamma^2\beta^2$,
since $\|e_i\|^2 = \|e'_i\|^2 = -1$ for $i = 1, 2, 3$ and $\gamma^2\beta_i^2 - u^2_i = \|e'_i\|^2$.

Lemma 3 :

If we consider ${}^t e_1' = (0, 1, 0, 0)$, ${}^t e_1' = (0, 0, 1, 0)$, ${}^t e_1' = (0, 0, 0, 1)$,
$f_1 = M e_1'$, $f_2 = M e_2'$, $f_3 = M e_3'$ *and*
P *the application* $P : {}^t(a, b, c, d) \to {}^t(b, c, d)$
then $\Omega = C^{-1}[P(f_1), P(f_2), P(f_3)]$ *and* $\vec{\beta}$ *is an eigenvector for* $C : C\vec{\beta} = \gamma\vec{\beta}$.

We have $M.\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^t(\gamma\vec{\beta})\,\Omega \\ C\Omega \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$ *and* $C\Omega = [P(f_1), P(f_2), P(f_3)]$.

As Maplesoft gives

$$det(C\Omega) = det(C) = \frac{\gamma^2\beta_1^2 + \gamma^2\beta_2^2 + \gamma^2\beta_3^2 + \gamma + 1}{1+\gamma} = 1 + \frac{\gamma^2\vec{\beta}\cdot\vec{\beta}}{1+\gamma} = 1 + \frac{\gamma^2 - 1}{1+\gamma} = \gamma \geq 1.$$

we have $\Omega = C^{-1}[P(f_1), P(f_2), P(f_3)]$

Maplesoft gives

$$C^{-1} = \begin{bmatrix} \frac{\gamma^2\beta_2^2 + \gamma^2\beta_3^2 + \gamma + 1}{\gamma\cdot(\gamma+1)} & -\frac{\gamma^2\beta_1\beta_2}{\gamma\cdot(\gamma+1)} & -\frac{\gamma^2\beta_1\beta_3}{\gamma\cdot(\gamma+1)} \\ -\frac{\gamma^2\beta_1\beta_2}{\gamma\cdot(\gamma+1)} & \frac{\gamma^2\beta_1^2 + \gamma^2\beta_3^2 + \gamma + 1}{\gamma\cdot(\gamma+1)} & -\frac{\gamma^2\beta_2\beta_3}{\gamma\cdot(\gamma+1)} \\ -\frac{\gamma^2\beta_1\beta_3}{\gamma\cdot(\gamma+1)} & -\frac{\gamma^2\beta_2\beta_3}{\gamma\cdot(\gamma+1)} & \frac{\gamma^2\beta_1^2 + \gamma^2\beta_2^2 + \gamma + 1}{\gamma\cdot(\gamma+1)} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\gamma - \gamma\beta_1^2 + 1}{1+\gamma} & -\frac{\gamma\beta_1\beta_2}{1+\gamma} & -\frac{\gamma\beta_1\beta_3}{1+\gamma} \\ -\frac{\gamma\beta_1\beta_2}{1+\gamma} & \frac{\gamma - \gamma\beta_2^2 + 1}{1+\gamma} & -\frac{\gamma\beta_2\beta_3}{1+\gamma} \\ -\frac{\gamma\beta_1\beta_3}{1+\gamma} & -\frac{\gamma\beta_2\beta_3}{1+\gamma} & \frac{\gamma - \gamma\beta_3^2 + 1}{1+\gamma} \end{bmatrix} = Id_{\mathbb{R}^3} - \frac{\gamma\vec{\beta}\,{}^t\vec{\beta}}{(1+\gamma)}$$

because $\gamma^2\beta_2^2 + \gamma^2\beta_3^2 + \gamma + 1 = \gamma^2\beta^2 - \gamma^2\beta_1^2 + \gamma + 1 = \gamma^2 - 1 - \gamma^2\beta_1^2 + \gamma + 1$
$= \gamma\left(1 + \gamma - \gamma\beta_1^2\right)$.

As $M = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta})\,\Omega \\ \gamma\vec{\beta} & C\Omega \end{bmatrix}$ *is a matrix of Lorentz :*

$$\begin{bmatrix} \gamma & \gamma \overset{t\leftrightarrow}{\beta} \\ \gamma\, {}^t\Omega\cdot\vec{\beta} & {}^t\Omega\cdot C \end{bmatrix} . \begin{bmatrix} 1 & 0 \\ 0 & -1_{\mathbb{R}^3} \end{bmatrix} . \begin{bmatrix} \gamma & \gamma \overset{\leftrightarrow}{\beta}\, \Omega \\ \gamma\vec{\beta} & C\,\Omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1_{\mathbb{R}^3} \end{bmatrix} \text{ then:}$$

$$\begin{bmatrix} -\beta\gamma^2 \overset{t\leftrightarrow}{\beta} + \gamma^2 & -C\,\Omega\gamma \overset{t\leftrightarrow}{\beta} + \Omega\gamma^2 \overset{t\leftrightarrow}{\beta} \\ -{}^t\Omega C\, \gamma\vec{\beta} + {}^t\Omega\gamma^2 \vec{\beta} & {}^t\Omega\gamma^2 \vec{\beta}\overset{\leftrightarrow}{\beta}\, \Omega - {}^t\Omega C^2\, \Omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1_{\mathbb{R}^3} \end{bmatrix} \text{ and}$$

$\gamma^2\left(1 - \vec{\beta}^2\right) = 1$;

${}^t\Omega\gamma^2 \vec{\beta}\overset{\leftrightarrow}{\beta}\, \Omega - {}^t\Omega C^2\, \Omega = -1_{\mathbb{R}^3} \Rightarrow {}^t\Omega\left(\gamma^2 \vec{\beta}\overset{\leftrightarrow}{\beta}{}^t - C^2\right)\Omega = -1_{\mathbb{R}^3} \Rightarrow C^2 = 1_{\mathbb{R}^3} + \gamma^2 \vec{\beta}\overset{\leftrightarrow}{\beta}$ since ${}^t\Omega\Omega$
$= 1_{\mathbb{R}^3}$;

${}^t\Omega C\, \gamma\vec{\beta} = {}^t\Omega\gamma^2 \vec{\beta} \Rightarrow C\vec{\beta} = \gamma\vec{\beta}$.

Note :
*For the rest, we recall that the change of basis matrix **M** has its columns equal to the expression of the basis vectors of the new base expressed in the old basis.*

Corollary :

(0) *If* M *is of Lorentz we have* M *symmetric* $\Leftrightarrow \Omega = Id_{\mathbb{R}^3}$.

With the uniqueness of the decomposition we have

$$M = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C \end{bmatrix} . \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} = M.Id_{\mathbb{R}^4}.$$

(1) *If* $M = M'.M''$ *is a product of two matrices de Lorentz with*

$$M = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta})\,\Omega \\ \gamma\vec{\beta} & C\Omega \end{bmatrix}, \; M' = \begin{bmatrix} \gamma' & {}^t(\overrightarrow{\gamma'\beta'}) \\ \overrightarrow{\gamma'\beta'} & C' \end{bmatrix}, \; M'' = \begin{bmatrix} \gamma'' & {}^t(\overrightarrow{\gamma''\beta''}) \\ \overrightarrow{\gamma''\beta''} & C'' \end{bmatrix}$$

$$\text{then} \quad M = \begin{bmatrix} \gamma'\gamma''\left(\overset{t\leftrightarrow}{\beta'}\vec{\beta}'' + 1\right) & \gamma'\gamma''\overset{t\longleftrightarrow}{\beta''} + \gamma'\overset{t\longleftrightarrow}{\beta'}C'' \\ \gamma'\gamma''\overrightarrow{\beta'} + \gamma''C'\overrightarrow{\beta''} & \gamma'\gamma''\vec{\beta}'\overset{\leftrightarrow}{\beta}'' + C'C'' \end{bmatrix} \Rightarrow \gamma = \gamma'\gamma''\left(\overset{\leftrightarrow}{\beta}'\vec{\beta}'' + 1\right),$$

$$\vec{\beta} = \frac{\gamma'\overrightarrow{\beta'} + C'\overrightarrow{\beta''}}{\gamma'\left(\overset{\leftrightarrow}{\beta}'\vec{\beta}'' + 1\right)}, \; C = Id_{\mathbb{R}^3} + \frac{\gamma^2\vec{\beta}\overset{\leftrightarrow}{\beta}}{(1+\gamma)} \text{ and } \Omega = C^{-1}\left(\gamma'\gamma''\beta'\,{}^t\beta'' + C'C''\right).$$

(2)

If we have $\Omega = Id_{\mathbb{R}^3}$ *and* $P(f_1) /\!/ P(e_1)$, $P(f_2) /\!/ P(e_2)$, $P(f_3) /\!/ P(e_3)$ *then* $\beta_i\beta_j = 0$ *for* $i \neq j$

$\Leftrightarrow \vec{\beta}$ *is parallel to an* e_i $i = 1, 2, 3$.

$$\Omega = Id_{\mathbb{R}^3} \Rightarrow C = [P(f_1), P(f_2), P(f_3)] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

since $P(f_1) /\!/ P(e_1)$, $P(f_2) /\!/ P(e_2)$, $P(f_3) /\!/ P(e_3)$,

$\Rightarrow C$ *is diagonal* $\Leftrightarrow \beta_i \beta_j = 0$ *for* $i \neq j \Leftrightarrow \vec{\beta}$ *is parallel to an* e_i $i = 1, 2, 3$.

(3) *If we have* $P(f_1) /\!/ P(e_1), P(f_2) /\!/ P(e_2), P(f_3) /\!/ P(e_3)$ *and*

$\vec{\beta}$ *is parallel to an* e_i $i = 1, 2, 3$ *then* $\Omega = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$:

$\vec{\beta}$ *is parallel to an* e_i $\Rightarrow C$ *is a diagonal matrix* ,

$P(f_1) /\!/ P(e_1), P(f_2) /\!/ P(e_2), P(f_3) /\!/ P(e_3) \Rightarrow C\Omega$ *is also a diagonal matrix*

$\Rightarrow \Omega$ *is diagonal* $\Rightarrow \Omega = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$ *since* Ω *is orthogonal* .

(4) *Let* M *be a matrix of Lorentz such* $\varepsilon = 1$ *and* $P(f_i) = \lambda_i P(e_i)$ *with* $\lambda_i > 0$ *for* $i = 1, 2, 3$,
then $\Omega = Id_{\mathbb{R}^3}$.

Let us consider $Q = \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}$ *with* ω *a rotation de* $\mathbb{R}^3$ *such as* $PQ(e_1) = \mu\vec{\beta}$, $\mu > 0$.

Since $P(f_i) = \lambda_i P(e_i)$ *with* $\lambda_i > 0$ *for* $i = 1, 2, 3$ *and* $\vec{\beta}$ *common to both basis* $\mathscr{B}$ *and* $\mathscr{B}'$
$PQ(e'_1) = \mu'\vec{\beta}$, $\mu' > 0$.

We can consider now :

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{M} & \mathscr{B}' \\ Q\downarrow & & \downarrow Q \\ \mathscr{B}_1 & \xrightarrow{M'} & \mathscr{B}'_1 \end{array}$$

with M' *a change of basis matrix of Lorentz.*

If we write $M = \Lambda\tilde{\Omega}$ *with* $\tilde{\Omega} = \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}$ *and* $\Lambda = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C \end{bmatrix}$:

$$M' = {}^tQMQ = {}^tQ\Lambda\tilde{\Omega}Q = ({}^tQ\Lambda Q)({}^tQ\tilde{\Omega}Q)$$

since M' *and* $({}^tQ\Lambda Q)$ *are symmetric and* ${}^tQ\tilde{\Omega}Q$ *orthogonal then* $\Omega = Id_{\mathbb{R}^3}$ *by uniqueness* .

Corollary :

Let M *be a change of basis matrix of Lorentz such* $\varepsilon = 1$ *associated to* **2** *observers* O *and* O':

$\mathscr{B} = (e_i)_{i=0,3} \xrightarrow{M} \mathscr{B}' = (e'_i)_{i=0,3}$ • *Let's consider another basis* $\mathscr{B}_1 = (f'_i)_{i=0,3}$ *associated to* O'**:**

$\mathscr{B}'$ *and* $\mathscr{B}_1$ *being associated to the same observer, the change of basis matrix is an orthogonal matrix* $\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\omega} \end{bmatrix}$ *with* ${}^t\boldsymbol{\omega\omega} = \boldsymbol{Id}_{\mathbb{R}3}$, $\mathscr{B}_1$ *chosen as* $\boldsymbol{P}(\boldsymbol{f_i}') = \lambda_i \boldsymbol{P}(\boldsymbol{e_i})$ *with* $\lambda_i > 0$ *for* $\boldsymbol{i} = \boldsymbol{1, 2, 3}$ *and* $\boldsymbol{P}$ *the application* $\boldsymbol{P} : {}^t(\boldsymbol{a, b, c, d}) \to {}^t(\boldsymbol{b, c, d})$:

Let's $\boldsymbol{M}'$ *the matrix of Lorentz from* $\mathscr{B}$ *to* $\mathscr{B}_1$:

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{\boldsymbol{M}} & \mathscr{B}' \\ \boldsymbol{M}' \searrow & & \nearrow \boldsymbol{Q} \\ & \mathscr{B}_1 & \end{array}$$

then $\boldsymbol{MX}' = \boldsymbol{M}'\boldsymbol{QX}'$ *with* $\boldsymbol{M}' = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & \boldsymbol{C} \end{bmatrix}$ *and* $\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\omega} \end{bmatrix}$ *with* ${}^t\boldsymbol{\omega\omega} = \boldsymbol{Id}_{\mathbb{R}3}$.

A special case :

Lemma 4 :

Let's assume that $\vec{\beta} = {}^t(\beta_1, 0, 0)$ *with* $\beta_1 \geq 0$ • *We put* $\delta_{i,j} = 1$ *if* $i = j$ *else* $\delta_{i,j} = 0$.

Let's consider $e_i = (\delta_{i,j})$, $j = 0, ..., 3$ *and* $M e_i$ *for* $i = 0, ..., 3$.

Let's put $\lceil {}^t(X_0, X_1, X_2, X_3) \rceil = {}^t(X_1, X_2, X_3)$.

We assume that $\lceil M e_i \rceil = \lambda_i \lceil e_i \rceil$ *with* $\lambda_i > 0$ *for* $i = 1, ..., 3$ (1),

we have :

$$M = \begin{bmatrix} \gamma & \gamma\beta_1 & 0 & 0 \\ \gamma\beta_1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Proof :

As $1 + \frac{\gamma^2}{(1+\gamma)} \beta^2{}_1 = \frac{(1+\gamma) + \gamma^2 \vec{\beta}^2}{(1+\gamma)} = \frac{1 + \gamma + \gamma^2 - 1}{(1+\gamma)} = \gamma$,

since $\gamma^2 - 1 = \frac{1}{1 - \vec{\beta}^2} - 1 = \frac{\vec{\beta}^2}{1 - \vec{\beta}^2} = \gamma^2 \vec{\beta}^2$ *we can write :*

$$M = exp(\alpha N) \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta_1 & 0 & 0 \\ \gamma\beta_1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \omega_{2,2} & \omega_{2,3} & \omega_{2,4} \\ 0 & \omega_{3,2} & \omega_{3,3} & \omega_{3,4} \\ 0 & \omega_{4,2} & \omega_{4,3} & \omega_{4,4} \end{bmatrix}.$$

$$M e_1 = \begin{bmatrix} \gamma\beta_1 \, \omega_{2,2} \\ \gamma\omega_{2,2} \\ \omega_{3,2} \\ \omega_{4,2} \end{bmatrix}, M e_2 = \begin{bmatrix} \gamma\beta_1 \, \omega_{2,3} \\ \gamma\omega_{2,3} \\ \omega_{3,3} \\ \omega_{4,3} \end{bmatrix} \text{ and } M e_3 = \begin{bmatrix} \gamma\beta_1 \, \omega_{2,4} \\ \gamma\omega_{2,4} \\ \omega_{3,4} \\ \omega_{4,4} \end{bmatrix}$$

then , by (1) : $\begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \omega_{2,2} & 0 & 0 \\ 0 & 0 & \omega_{3,3} & 0 \\ 0 & 0 & 0 & \omega_{4,4} \end{bmatrix}$.

As ${}^t\Omega\Omega = Id_{\mathbb{R}^3}$, $M = \begin{bmatrix} \gamma & \gamma\beta_1 & 0 & 0 \\ \gamma\beta_1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & \varepsilon_3 \end{bmatrix}$ with $\varepsilon_i = \pm 1$.

As $\lambda_i > 0$ and $\gamma > 0$ then $\varepsilon_i = 1$.

We can check ${}^t\mathbf{M}\mathbf{G}\mathbf{M} = \mathbf{G}$ and $|\mathbf{Me}_i| = \lambda_i |\mathbf{e}_i|$ **with** $\lambda_i > 0$ for $\mathbf{i} = \mathbf{1}, \ldots, \mathbf{3}$.

In the case where $\boldsymbol{\varepsilon} = \mathbf{1}$ and $|\mathbf{Me}_i| = \lambda_i |\mathbf{e}_i|$ with $\lambda_i > 0$ for $\mathbf{i} = \mathbf{1}, \ldots, \mathbf{3}$,

$\beta_1 \geq 0$, we have :

$$M = \begin{bmatrix} \gamma & \gamma\beta_1 & 0 & 0 \\ \gamma\beta_1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Another case :

Lemma 5 :

Let $\mathbf{M}$ be a matrix of Lorentz such $\boldsymbol{\varepsilon} = \mathbf{1}$ and $|\mathbf{Me}_i| = \lambda_i |\mathbf{e}_i|$ with $\lambda_i > 0$ for $\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}$,

for any $\vec{\beta}$ we have $\Omega = Id_{\mathbb{R}3}$ and $M = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C \end{bmatrix}$ with

$$C = \begin{bmatrix} 1 + \frac{\gamma^2}{(1+\gamma)}\beta_1^2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_1 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta_2^2 & \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_1 & \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_2 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta_3^2 \end{bmatrix}.$$

Proof :

Let's consider $P = \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}$ such ${}^t\Omega\Omega = Id_{\mathbb{R}3}$ and $P \begin{bmatrix} 0 \\ \vec{\beta} \end{bmatrix} = \lambda e_1$ with $\lambda > 0$

and $P\begin{bmatrix} 0 \\ \vec{\beta} \end{bmatrix} = \lambda' e'_1$ since $\vec{\beta}$ is common to both O and O'

and $|Me_1| = \lambda_1 |e_1|$ with $\lambda_1 > 0$.

We can consider now :

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{M} & \mathscr{B}' \\ P\downarrow & & \downarrow P \\ \mathscr{B}_1 & \xrightarrow{M'} & \mathscr{B}'_1 \end{array}$$

With the basis $\mathscr{B}_1 = \left(Pe_0 = e_0, Pe_1, Pe_2\,,\ Pe_3 \right)$

and $\mathscr{B}'_1 = \left(Pe'_0 = e'_0, Pe'_1, Pe'_2\,,\ Pe'_3 \right)$ M is represented by $M' = PM\,{}^tP$.

We have $M'Pe_i = PM\,{}^tPPe_i = PM\,e_i = \lambda_i Pe_i$ for $i = 1, 2, 3$.

$${}^tPGP = \begin{bmatrix} 1 & 0 \\ 0 & {}^t\Omega \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -Id_{\mathbb{R}3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} = G \text{ therefore}$$

M' is of Lorentz .

If $W = ct\begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$, $PW = ct\begin{bmatrix} 1 \\ \\ \Omega\vec{\beta} \\ \\ \end{bmatrix} = ct\begin{bmatrix} 1 \\ \beta'_1 \\ \beta'_2 \\ \beta'_3 \end{bmatrix} = ct\begin{bmatrix} 1 \\ \lambda \\ 0 \\ 0 \end{bmatrix}$ and then

$\vec{\beta'} = {}^t\left(\beta'_1, 0\,, 0\right)$ with $\beta_1 \geq 0$.

The lemma 4 shows that $M' = \begin{bmatrix} \gamma & \gamma\beta'_1 & 0 & 0 \\ \gamma\beta'_1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

If $\Lambda(\vec{\beta}) = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C \end{bmatrix}$ and $\overset{\curlywedge}{\Omega} = \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}$ are the factors

of the polar decomposition of M (lemma 3),

$M' = {}^tPMP = {}^tP\left(\Lambda(\vec{\beta})\,\overset{\curlywedge}{\Omega}\right)P = \left({}^tP\Lambda(\vec{\beta})P\right)\left({}^tP\overset{\curlywedge}{\Omega}P\right)$.

M' is symmetric, ${}^tP\Lambda(\vec{\beta})P$ is symmetric and ${}^tP\overset{\curlywedge}{\Omega}P$ is orthogonal since

${}^t\left({}^tP\overset{\curlywedge}{\Omega}P\right)\left({}^tP\overset{\curlywedge}{\Omega}P\right) = Id_{\mathbb{R}4}$. For short $M' = M' \cdot Id_{\mathbb{R}^4} = \left({}^tP\Lambda(\vec{\beta})P\right)\left({}^tP\overset{\curlywedge}{\Omega}P\right)$.

As the decomposition is unique ${}^tP\overset{\curlywedge}{\Omega}P = Id_{\mathbb{R}4}$ (and $M' = \left({}^tP\Lambda(\vec{\beta})P\right)$) .

and then $\overset{\curlywedge}{\Omega} = Id_{\mathbb{R}^4}$.

Eigenvalues and eigenvectors when $\Omega = Id_{\mathbb{R}^3}$.

Lemma 4 :

*With the lemma **1** notations :*

$$N = \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 \end{bmatrix} \; avec \, \vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \; et \, \vec{X}.\vec{X} = 1 \, .$$

As the eigenvalues of N are $\{1, -1, 0, 0\}$ and the eigenvectors are :

$$\left\{ \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} -1 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} 0 \\ -x_2 \\ x_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -x_3 \\ 0 \\ x_1 \end{bmatrix} \right\}$$

*If we consider **B** the basis made of thes eigenvectors and the diagonal matrix made of the eigenvalues of N : $N = BDB^{-1}$ • Don't forget that $x_1^2 + x_2^2 + x_3^2 = \vec{X}.\vec{X} = 1$.*

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ x_1 & x_1 & -x_2 & -x_3 \\ x_2 & x_2 & x_1 & 0 \\ x_3 & x_3 & 0 & x_1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 1 & -1 & 0 & 0 \\ x_1 & x_1 & -x_2 & -x_3 \\ x_2 & x_2 & x_1 & 0 \\ x_3 & x_3 & 0 & x_1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0 & \dfrac{x_1}{x_1^2 + x_2^2 + x_3^2} & \dfrac{x_2}{x_1^2 + x_2^2 + x_3^2} & \dfrac{x_3}{x_1^2 + x_2^2 + x_3^2} \\ x_1 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 \end{bmatrix}$$

Let's consider now $\exp(\alpha N)$, as $\exp(\alpha N) = \exp(B\alpha DB^{-1}) = B\exp(\alpha D)B^{-1}$ and

thus :

$$\exp(\alpha N) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ x_1 & x_1 & -x_2 & -x_3 \\ x_2 & x_2 & x_1 & 0 \\ x_3 & x_3 & 0 & x_1 \end{bmatrix} . \begin{bmatrix} e^{\alpha} & 0 & 0 & 0 \\ 0 & e^{-\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & -1 & 0 & 0 \\ x_1 & x_1 & -x_2 & -x_3 \\ x_2 & x_2 & x_1 & 0 \\ x_3 & x_3 & 0 & x_1 \end{bmatrix}^{-1}$$

However $\gamma = ch(\alpha) \Leftrightarrow \alpha = ln\left(\gamma + \sqrt{\gamma^2 - 1}\right) \Rightarrow e^{\alpha} = \gamma + \sqrt{\gamma^2 - 1,}$

as $\gamma^2 - 1 = \beta^2\gamma^2 \Rightarrow e^{\alpha} = \gamma(1+\beta) = \dfrac{(1+\beta)}{\sqrt{(1+\beta)(1-\beta)}} = \sqrt{\dfrac{1+\beta}{1-\beta}}$ *and*

$e^{-\alpha} = \gamma(1-\beta) = \sqrt{\dfrac{1-\beta}{1+\beta}}$ *with* $\beta = \sqrt{\vec{\beta}\cdot\vec{\beta}}$.

We assume that $\beta \neq 0$ *if not* $exp(\alpha N) = Id$.

As $coth(\alpha) = \beta^{-1}$ *and* $\vec{X} = coth(\alpha)\vec{\beta}$ *:*

We assume that $\beta \neq 0$

$$exp(\alpha N) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ \beta^{-1}\cdot\beta_1 & \beta^{-1}\cdot\beta_1 & -\beta^{-1}\cdot\beta_2 & -\beta^{-1}\cdot\beta_3 \\ \beta^{-1}\cdot\beta_2 & \beta^{-1}\cdot\beta_2 & \beta^{-1}\cdot\beta_1 & 0 \\ \beta^{-1}\cdot\beta_3 & \beta^{-1}\cdot\beta_3 & 0 & \beta^{-1}\cdot\beta_1 \end{bmatrix} \cdot \begin{bmatrix} \gamma\cdot(1+\beta) & 0 & 0 & 0 \\ 0 & \gamma\cdot(1-\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ \beta^{-1}\cdot\beta_1 & \beta^{-1}\cdot\beta_1 & -\beta^{-1}\cdot\beta_2 & -\beta^{-1}\cdot\beta_3 \\ \beta^{-1}\cdot\beta_2 & \beta^{-1}\cdot\beta_2 & \beta^{-1}\cdot\beta_1 & 0 \\ \beta^{-1}\cdot\beta_3 & \beta^{-1}\cdot\beta_3 & 0 & \beta^{-1}\cdot\beta_1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \gamma & \dfrac{\gamma\beta_1\beta^2}{\beta_1^2+\beta_2^2+\beta_3^2} & \dfrac{\gamma\beta^2\beta_2}{\beta_1^2+\beta_2^2+\beta_3^2} & \dfrac{\gamma\beta_3\beta^2}{\beta_1^2+\beta_2^2+\beta_3^2} \\ \beta_1\gamma & \dfrac{\gamma\beta_1^2+\beta_2^2+\beta_3^2}{\beta_1^2+\beta_2^2+\beta_3^2} & \dfrac{\beta_2\beta_1(\gamma-1)}{\beta_1^2+\beta_2^2+\beta_3^2} & \dfrac{\beta_3\beta_1(\gamma-1)}{\beta_1^2+\beta_2^2+\beta_3^2} \\ \beta_2\gamma & \dfrac{\beta_2\beta_1(\gamma-1)}{\beta_1^2+\beta_2^2+\beta_3^2} & \dfrac{\gamma\beta_2^2+\beta_1^2+\beta_3^2}{\beta_1^2+\beta_2^2+\beta_3^2} & \dfrac{\beta_2\beta_3(\gamma-1)}{\beta_1^2+\beta_2^2+\beta_3^2} \\ \beta_3\gamma & \dfrac{\beta_3\beta_1(\gamma-1)}{\beta_1^2+\beta_2^2+\beta_3^2} & \dfrac{\beta_2\beta_3(\gamma-1)}{\beta_1^2+\beta_2^2+\beta_3^2} & \dfrac{\gamma\beta_3^2+\beta_1^2+\beta_2^2}{\beta_1^2+\beta_2^2+\beta_3^2} \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & \gamma\cdot\beta_1 & \gamma\cdot\beta_2 & \gamma\cdot\beta_1 \\ \gamma\cdot\beta_1 & 1+\dfrac{\beta_1^2(\gamma-1)}{\beta^2} & \dfrac{\beta_2\beta_1(\gamma-1)}{\beta^2} & \dfrac{\beta_3\beta_1(\gamma-1)}{\beta^2} \\ \gamma\cdot\beta_2 & \dfrac{\beta_2\beta_1(\gamma-1)}{\beta^2} & 1+\dfrac{\beta_2^2(\gamma-1)}{\beta^2} & \dfrac{\beta_2\beta_3(\gamma-1)}{\beta^2} \\ \gamma\cdot\beta_3 & \dfrac{\beta_3\beta_1(\gamma-1)}{\beta^2} & \dfrac{\beta_2\beta_3(\gamma-1)}{\beta^2} & 1+\dfrac{\beta_3^2(\gamma-1)}{\beta^2} \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & \gamma\cdot\beta_1 & \gamma\cdot\beta_2 & \gamma\cdot\beta_3 \\ \gamma\cdot\beta_1 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_3 \\ \gamma\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_1 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_3 \\ \gamma\cdot\beta_3 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_2 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_3 \end{bmatrix}$$

because $1 \Big/ \left((1+\gamma)\,\gamma^{2} \right) = \dfrac{(\gamma-1)}{\beta^2}$.

We can verify :

$$\begin{bmatrix} \gamma & \gamma\cdot\beta_1 & \gamma\cdot\beta_2 & \gamma\cdot\beta_3 \\ \gamma\cdot\beta_1 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_3 \\ \gamma\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_1 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_3 \\ \gamma\cdot\beta_3 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_2 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \beta^{-1}\cdot\beta_1 \\ \beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_3 \end{bmatrix} =$$

$$\begin{bmatrix} \dfrac{\gamma\left(\beta_1^2+\beta_2^2+\beta_3^2+\beta\right)}{\beta} \\ \dfrac{\beta_1\left(1+\left(\beta_1^2+\beta_2^2+\beta_3^2+\beta\right)\gamma^2+(\beta+1)\gamma\right)}{(1+\gamma)\beta} \\ \dfrac{\beta_2\left(1+\left(\beta_1^2+\beta_2^2+\beta_3^2+\beta\right)\gamma^2+(\beta+1)\gamma\right)}{(1+\gamma)\beta} \\ \dfrac{\beta_3\left(1+\left(\beta_1^2+\beta_2^2+\beta_3^2+\beta\right)\gamma^2+(\beta+1)\gamma\right)}{(1+\gamma)\beta} \end{bmatrix} = \gamma(1+\beta)\begin{bmatrix} 1 \\ \beta^{-1}\cdot\beta_1 \\ \beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_3 \end{bmatrix} \text{ because}$$

$$1+\left(\beta_1^2+\beta_2^2+\beta_3^2+\beta\right)\gamma^2+(\beta+1)\gamma = 1+\beta\cdot(1+\beta)\gamma^2+(\beta+1)\gamma$$
$$=1+(\beta+1)\gamma(1+\gamma).$$

$$\text{thus } \frac{\beta_1\left(1+\left(\beta_1^2+\beta_2^2+\beta_3^2+\beta\right)\gamma^2+(\beta+1)\gamma\right)}{(1+\gamma)\beta} = \frac{\beta_1(1+(\beta+1)\gamma(1+\gamma))}{(1+\gamma)\beta} \text{ however}$$

$$\gamma(1+\beta)\beta^{-1}\cdot\beta_1 = \frac{\beta_1\left(1+\left(\beta^2+\beta\right)\gamma^2+(\beta+1)\gamma\right)}{(1+\gamma)\beta} \Leftrightarrow \gamma(1+\beta)\cdot\beta_1(1+\gamma) = \beta_1\left(1+\beta\left(\gamma+\gamma^2\right)+\gamma^2-1+\gamma\right)$$

because $\gamma^2\beta^2=\gamma^2-1$ et $\beta_1\left(\beta\left(\gamma+\gamma^2\right)+\gamma^2+\gamma\right)=\beta_1\left(\gamma+\gamma^2\right)(1+\beta)=\beta_1(1+\gamma)\gamma(1+\beta)$.

In the same way :

$$\begin{bmatrix} \gamma & \gamma\cdot\beta_1 & \gamma\cdot\beta_2 & \gamma\cdot\beta_3 \\ \gamma\cdot\beta_1 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_3 \\ \gamma\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_1 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_3 \\ \gamma\cdot\beta_3 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_2 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_3 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ \beta^{-1}\cdot\beta_1 \\ \beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_3 \end{bmatrix} =$$

$$\begin{bmatrix} -\dfrac{\gamma\left(-\beta_1^2-\beta_2^2-\beta_3^2+\beta\right)}{\beta} \\ -\dfrac{\left(-1+\left(-\beta_1^2-\beta_2^2-\beta_3^2+\beta\right)\gamma^2+(\beta-1)\gamma\right)\beta_1}{(1+\gamma)\beta} \\ -\dfrac{\left(-1+\left(-\beta_1^2-\beta_2^2-\beta_3^2+\beta\right)\gamma^2+(\beta-1)\gamma\right)\beta_2}{(1+\gamma)\beta} \\ -\dfrac{\left(-1+\left(-\beta_1^2-\beta_2^2-\beta_3^2+\beta\right)\gamma^2+(\beta-1)\gamma\right)\beta_3}{(1+\gamma)\beta} \end{bmatrix} = \gamma(1-\beta)\begin{bmatrix} -1 \\ \beta^{-1}\cdot\beta_1 \\ \beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_3 \end{bmatrix}.$$

We can point out that the 2 first eigenvectors are light vectors .

$$\begin{bmatrix} \gamma & \gamma\cdot\beta_1 & \gamma\cdot\beta_2 & \gamma\cdot\beta_3 \\ \gamma\cdot\beta_1 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_3 \\ \gamma\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_1 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_3 \\ \gamma\cdot\beta_3 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_2 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -\beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_1 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} \gamma & \gamma\cdot\beta_1 & \gamma\cdot\beta_2 & \gamma\cdot\beta_3 \\ \gamma\cdot\beta_1 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_3 \\ \gamma\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_1 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_3 \\ \gamma\cdot\beta_3 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_2 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -\beta^{-1}\cdot\beta_3 \\ 0 \\ \beta^{-1}\cdot\beta_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\beta^{-1}\cdot\beta_3 \\ 0 \\ \beta^{-1}\cdot\beta_1 \end{bmatrix}.$$

For short the eigenvalues are $\{\gamma(1+\beta), \gamma(1-\beta), 1, 1\}$ *and*

the eingeinvectors $\left\{ \begin{bmatrix} 1 \\ \beta^{-1}\cdot\beta_1 \\ \beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_3 \end{bmatrix}, \begin{bmatrix} -1 \\ \beta^{-1}\cdot\beta_1 \\ \beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_3 \end{bmatrix}, \begin{bmatrix} 0 \\ -\beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\beta^{-1}\cdot\beta_3 \\ 0 \\ \beta^{-1}\cdot\beta_1 \end{bmatrix} \right\}$

Eigenvalues of a product of 2 symmetric matrices :
If $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are real symmetric matrices such that the eigenvalues of* $\boldsymbol{A}$ *are strictly positive and those of* $\boldsymbol{B}$ *positive then the product* $\boldsymbol{AB}$ *is also* $\mathbb{R}-$*diagonalizable .*

(*We remind that if* $\boldsymbol{A}$ *is symmetric with positive eigenvalue then there exists a single matrice symmetric with eigenvalues positive* $A^{\frac{1}{2}}$ *such as* $\left(A^{\frac{1}{2}} \bullet A^{\frac{1}{2}}\right) = A$).

Proof :

We can write $AB = A^{\frac{1}{2}}\left(A^{\frac{1}{2}} B A^{\frac{1}{2}}\right)A^{-\frac{1}{2}}$ *, thus* $\boldsymbol{AB}$ ***and*** $A^{\frac{1}{2}} B A^{\frac{1}{2}}$ *have the same eigenvalues .*

However ${}^{t}\left(A^{\frac{1}{2}} B A^{\frac{1}{2}}\right) = \left(A^{\frac{1}{2}} B A^{\frac{1}{2}}\right)$ *thus* $\left(A^{\frac{1}{2}} B A^{\frac{1}{2}}\right)$ *is symmetric therefore* $\mathbb{R}$-*diagonalizable .*

Note : If $\boldsymbol{\Omega} \neq \boldsymbol{Id}_{\mathbb{R}3}$ $\boldsymbol{M}$ *may have no real eigenvalues for example* $\vec{\beta} = \vec{0}$ *and* $\boldsymbol{\Omega}$ *being a rotation .*

(5) **Space − time vectors properties :**

*In a Minkowski space, the quadratic form associated with this space makes it possible to classify the vectors into **3** categories that we are going to study.*

Vectors in a Minkowski − space

(*J - M. Souriau.*"Calcul Linéaire " • *PUF 1964* .)

We consider the vector space $\mathbb{R}^4$ *of* **4** *dimensions with the quadratic form of Lorentz :*

$$\Phi(X) = x_1^2 - \sum_{i=2}^{4} x_i^2 \text{ where } {}^bX = (x_1, \ldots, x_4) \text{, whose matrix is :}$$

$$G = \begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbb{R}^3} \end{bmatrix} \text{ where } 0 \text{ is the null column of } \mathbb{R}^3.$$

We give $\mathbb{R}^4$ *a* Φ *- orthonormal basis .*

We want to describe the sets of vectors $\mathbb{R}^4$ *defined according **to** the sign of* $\Phi(X)$.

We distinguish 3 subsets :

$E_+ = \{X \in \mathbb{R}^4 / \Phi(X) > 0\} \cup \{0\}$ *: time vectors,*

$E_- = \{X \in \mathbb{R}^4 / \Phi(X) < 0\} \cup \{0\}$ *: space vectors,*

$E_0 = \{X \in \mathbb{R}^4 / \Phi(X) = 0\}$ *: isotropic vectors .*

Whether or not to add $\{0\}$ *to* E_+ *and to* E_- *varies according to different authors.*

We notice that the nature of vectors is independent of the chosen basis of $\mathbb{R}^4$ Φ *- orthonormed .*
If we note P *the matrix of passage from the basis* $\mathcal{B}$ *to the basis* $\mathcal{B}'$,
if P *is a matrix of Lorentz;* ${}^tPGP = G$, $X = PX'$, $Y = PY'$ *then*

$${}^tX\,GX = ({}^tX'\,{}^tP)\,G(PX') = {}^tX'({}^tPGP)X' = {}^tX'GX'.$$

Lemma 1 :

We consider the vector space $\mathbb{R}^4$ *of* **4** *dimensions with the quadratic form of Lorentz.*
In $\mathbb{R}^4$ *there is a* **3** *− dimensional subspace of space vectors .*

Proof :

Consider for example : $\{X \in \mathbb{R}^4 / x_1 = 0\}$.

Lemma 2 :

We consider the vector space $\mathbb{R}^4$ *with the quadratic form of Lorentz.*
There are no time vectors subspace of dimension $> = 2$.

Proof :

Because otherwise there exists a subspace F *of dimension at least equal to* **2** *of time vectors .*
As there exists a subspace G of space vectors of dimension **3** *and as* $F \cap G = \{0\}$ *and therefore* $\dim(\mathbb{R}^4) \geq 5$.

Lemma 3 :

We consider the vector space with the quadratic form of Lorentz.
Let **2** *vectors of* $\mathbb{R}^4$ X *and* $Y \neq \{0\}$ *such that* ${}^tXGX \geq 0$, ${}^tYGY \geq 0$ *and* ${}^tXGY = 0$
then X *and* Y *are collinear and isotropic.*

Proof

We consider X *and* Y **2** *non - zero time vectors.*
If they are independent they generate a subspace of dimension **2** *of positive vectors :*

If $Z = \lambda X + \mu Y$ *then* ${}^tZGZ = {}^t(\lambda X + \mu Y)\,G(\lambda X + \mu Y) = \lambda^2\,{}^tXGX + \mu^2\,{}^tYGY \geq 0$.

It is impossible therefore : $\exists\, \lambda \neq 0 \in \mathbb{R}$ *such that*

$X=\lambda Y$ and then $: 0={}^tXGY=\lambda {}^tY\,GY=\frac{1}{\lambda}{}^tX\,GX$.

Lemma 4 :

(1) Any non $-$ zero vector X orthogonal to a non - zero time vector Y $\left({}^tXGY=0\right)$ is a space vector.
(2) 2 non $-$ zero isotropic independent vectors X and Y are never orthogonal.

Proof :

(1) If X was a non $-$ zero time vector : ${}^tXGX>0$ and Y such that ${}^tXGX>0$ then ${}^tYGY<0$ because otherwise ${}^tYGY\geq 0$ **and** as ${}^tXGX>0$ and ${}^tXGY=0$, according to **lemma 3** X would be isotropic, which is contradictory.

(2) Because otherwise $0={}^tXGY={}^tXGX={}^tYGY$ according to **lemma 5** X and Y are linearly dependent.

Definition : For any time vector $\left({}^tXGX\geq 0\right)$ we set $\|X\|_G=\sqrt{{}^tXGX}$.

Lemma 5 :

Let φ be a bilinear symmetrical nondegenerate form on a vector space E of dimension n, then for any base $\mathscr{B}\left(e_1, ..., e_n\right)$, if we consider the matrix representing φ in $\mathscr{B}$

$Q=\left(\varphi\left(e_i, e_j\right)\right)$, then the determinant of Q is of the sign of $(-1)^{n-p}$ where p is the positive index of inertia of φ.

Proof :

There exists a base $\mathscr{B}'\left(e'_1, ..., e'_n\right)$ where the matrix representing φ is the form :

$$Q'=\begin{bmatrix} Id_{\mathbb{R}^p} & 0 \\ 0 & -Id_{\mathbb{R}^q}\end{bmatrix}, \text{then } \det(Q')=(-1)^{n-p}.$$

Let S be the matrix of passage from the basis $\mathscr{B}'$ to the basis $\mathscr{B}$ we have

$\det(Q)=\det\left({}^tSQ'S\right)=\det(Q')\left(\det(S)\right)^2 \Rightarrow \operatorname{sign}\left(\det(Q)\right)=(-1)^{n-p}$.

Lemma 6 : Cauchy $-$ Schwartz's counter inequality.

We have $\left|{}^tXGY\right| \geq \|X\|_G\|Y\|_G$ for any time or isotropic vectors.

Proof :

Let us consider the matrix S made up of the 2 columns $X\in\mathbb{R}^4$ and $Y\in\mathbb{R}^4$: $S=[X, Y]$.
We assume that X and Y are non $-$ zero time vectors because if one of them is isotropic or zero the `inequality is obvious.

$$\text{We have } {}^tSGS=\begin{bmatrix} {}^tX \\ {}^tY\end{bmatrix} G[X, Y]=\begin{bmatrix} {}^tXGX & {}^tXGY \\ {}^tYGX & {}^tYGY\end{bmatrix},$$

and $\det\left({}^tSGS\right)=\|X\|_G^2\|Y\|_G^2-\left({}^tXGY\right)^2$.

If $\det\left({}^tSGS\right)=0$ the lemma is proved.
If $\det\left({}^tSGS\right)\neq 0$ and if $X=\lambda Y$, $\lambda\in\mathbb{R}^*$,

$$\text{then } \det\left({}^tSGS\right)=\begin{vmatrix} \lambda^2\,{}^tYGY & \lambda\,{}^tYGY \\ \lambda\,{}^tYGY & {}^tYGY\end{vmatrix}=0$$

It's impossible therefore X and Y are non coplanar : X and Y form a base $S=[X, Y]$ of a vector subspace F of $\mathbb{R}^4$ of dimension 2.

As $^{t}SGS = \begin{bmatrix} {}^{t}XGX & {}^{t}XGY \\ {}^{t}YGX & {}^{t}YGY \end{bmatrix}$, ^{t}SGS defines a form, bilinear and symmetrical Ψ

on F by $\Psi(u, v) = {}^{t}U\, {}^{t}SGSV$ with $u = SU$ and $v = SV$.
Let φ be thebilinear form defined` on $\mathbb{R}^4$ whose matrix is G.
We immediately check that $\varphi_{/F}$ is also a bilinear symmetrical form on F.

Let us show that $\varphi_{/F}$ is regular, that is :

$F^{\perp} = \{u \in F / \varphi_{/F}(u, v) = 0 \ , \forall v \in F\} = \{0\}$.

S, being a basis of F, is a bijective application of $\mathbb{R}^2$ on F.

If G is the representation of φ $\mathbb{R}^4$, ^{t}SGS is the representation of $\varphi_{/F}$ in F

provided with the base S : let u and v 2 vectors of F;

we have if $\tilde{U}$ and $\tilde{V}$ the representations of u and v in $\mathbb{R}^4$,

U and V in F with the base S and $\tilde{U} = SU$, $\tilde{V} = SV$:

$\varphi(u, v) = \varphi_{/F}(u, v) = {}^{t}\tilde{U}\, G\, \tilde{V} = {}^{t}U\, {}^{t}SGSV.$

So let u $u \in F^{\perp}$ and $v \in F$, $u = SU$ and $v = SV$ with U and V element of $\mathbb{R}^2$.

We have $\varphi_{/F}(u, v) = {}^{t}V\, {}^{t}SGSU = 0 \quad \forall V \in \mathbb{R}^2$ and therefore $^{t}SGSU = 0$

as $\det(SGS) \neq 0$ we have $U = 0$.

$\varphi_{/F}$ is indeed a bilinear symmetric regular form on F.

The **lemma 5** applies : $\operatorname{sign}(\det({}^{t}SGS)) = (-1)^{2-p}$, p the inertia index of $\varphi_{/F}$

but F being of dimension 2, according to the **lemma 2** there is no subspace of dimension $>= 2$ of time vectors

and F contains X, time vector.

$\varphi_{/F}$ being a bilinear symmetrical regular form on F, $\varphi_{/F}$ can be represented

in a $\varphi_{/F}$ – orthogonal basis by a diagonal matrix composed` of 1, -1 and 0.

The nature of the vectors remaining unchanged, the only possibility is therefore 1 and -1.
So the only possibility for p is $p = 1$.

Therefore $\operatorname{sign}(\det({}^{t}SGS)) = -1$ **et** donc $\|X\|_G^2 \|Y\|_G^2 < ({}^{t}XGY)^2$.

Note;

We recall that the set of time vectors do not form a vector subspace
take for example; $X = {}^{t}(4, 1, 1, 1)$ **et** $Y = {}^{t}(-4, 1, 1, 1)$ **et** $X + Y = {}^{t}(0, 2, 2, 2)$.

***Lemme** 7 :*

Let X, Y and Z 3 time or isotrope - vectors we then have :

$$({}^{t}XGY)({}^{t}YGZ)({}^{t}ZGX) \geq ({}^{t}XGX)({}^{t}YGY)({}^{t}ZGZ)$$

Proof:

Let's consider $S = [X, Y, Z]$, we have :

$$ {}^tSGS = \begin{bmatrix} {}^tX \\ {}^tY \\ {}^tZ \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -Id_{\mathbb{R}^3} \end{bmatrix} [X, Y, Z] = \begin{bmatrix} X_1 & -X_2 & -X_3 & -X_4 \\ Y_1 & -Y_2 & -Y_3 & -Y_3 \\ Z_1 & -Z_2 & -Z_3 & -Z_4 \end{bmatrix} \begin{bmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \\ X_4 & Y_3 & Z_4 \end{bmatrix} $$

$$ = \begin{bmatrix} {}^tXGX & {}^tXGY & {}^tXGZ \\ {}^tYGX & {}^tYGY & {}^tYGZ \\ {}^tZGX & {}^tZGY & {}^tZGZ \end{bmatrix} $$

and $\det({}^tSGS) = {}^tXGX[{}^tYGY \cdot {}^tZGZ - {}^tYGZ \cdot {}^tZGY]$

$$ -{}^tXGY[{}^tYGX \cdot {}^tZGZ - {}^tYGZ \cdot {}^tZGX] $$

$$ + {}^tXGZ[{}^tYGX \cdot {}^tZGY - {}^tYGY \cdot {}^tZGX] $$

$$ = {}^tXGX \cdot {}^tYGY \cdot {}^tZGZ - {}^tXGX \cdot {}^tYGZ \cdot {}^tZGY $$

$$ -{}^tXGY \cdot {}^tYGX \cdot {}^tZGZ + {}^tXGY \cdot {}^tYGZ \cdot {}^tZGX $$

$$ + {}^tXGZ \cdot {}^tYGX \cdot {}^tZGY - {}^tXGZ \cdot {}^tYGY \cdot {}^tZGX $$

$$ = {}^tXGX \cdot {}^tYGY \cdot {}^tZGZ - {}^tXGX({}^tYGZ)^2 $$

$$ -{}^tYGY \cdot ({}^tXGZ)^2 - {}^tZGZ\,({}^tXGY)^2 + 2 \cdot {}^tXGY \cdot {}^tYGZ \cdot {}^tZGX ; $$

But $|{}^tXGY| \geq \sqrt{{}^tXGX}\sqrt{{}^tYGY}$, $|{}^tYGZ| \geq \sqrt{{}^tYGY}\sqrt{{}^tZGZ}$, $|{}^tZGX| \geq \sqrt{{}^tZGZ}\sqrt{{}^tXGX}$, *then :*

$$ -{}^tXGX({}^tYGZ)^2 \; -{}^tYGY \cdot ({}^tXGZ)^2 - {}^tZGZ\,({}^tXGY)^2 $$

$$ \leq -{}^tXGX \cdot {}^tYGY \cdot {}^tZGZ - {}^tYGY \cdot {}^tZGZ \cdot {}^tXGX - {}^tZGZ \cdot {}^tXGX \cdot {}^tYGY $$

$$ = -3 \cdot {}^tXGX \cdot {}^tYGY \cdot {}^tZGZ \, , $$

and then $\det({}^tSGS) \leq -2 \cdot {}^tXGX \cdot {}^tYGY \cdot {}^tZGZ + 2{}^tXGY \cdot {}^tYGZ \cdot {}^tZGX$.

Considering the sign of $\det({}^tSGS)$ *:*

If $\det({}^tSGS) = 0$ *the lemma is proved .*

If $\det({}^tSGS) \neq 0$ *and if* $X = \lambda Y + \mu Z$, $\lambda \in \mathbb{R}^*$, $\mu \in \mathbb{R}^*$,

we have $\det({}^tSGS) = \det \begin{bmatrix} {}^tXGX & {}^tXGY & {}^tXGZ \\ {}^tYGX & {}^tYGY & {}^tYGZ \\ {}^tZGX & {}^tZGY & {}^tZGZ \end{bmatrix} = 0$ *because* $(\lambda^tY + \mu^tZ)GX = \lambda^tYGX + \mu^tZGX$

it's impossible therefore X , Y *et* Z *are non* $-$ *collinear and form the basis of a vector subspace F of* $\mathbb{R}^4$ *of dimension* 3.

As ${}^tSGS = \begin{bmatrix} {}^tXGX & {}^tXGY & {}^tXGZ \\ {}^tYGX & {}^tYGY & {}^tYGZ \\ {}^tZGX & {}^tZGY & {}^tZGZ \end{bmatrix}$, ${}^tSGS$ *is a bilinear symetrical form on* F.

By making a similar reasoning ***to*** *that of the previous lemma we find that* $sign(\det({}^tSGS)) = (-1)^{3-1} = 1$.

Therefore $\det({}^tSGS) > 0$ *and the lemma is proved.*

Lemma 8 :

We consider $\mathbb{R}^4$ *provided with the quadratic form defined by* $G = \begin{bmatrix} 1 & 0 \\ 0 & -Id_{\mathbb{R}^3} \end{bmatrix}$,

then the union of non − zero time vectors and the isotropic vectors are divided into **2** *opposite classes* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *and if* **X** *and* **Y** *are non -zero time vectors we have :*

$${}^tXGY \geq 0 \Leftrightarrow X \text{ and } Y \text{ belong to the same class },$$
$${}^tXGY \leq 0 \Leftrightarrow X \text{ and } Y \text{ belong to opposite classes }.$$

Two vectors **X** *and* **Y** *belonging to a same class check* ${}^tXGY > 0$ *unless they are isotropic and parallel. In this case they belong to the same class if their ratio is a strictly positive number.*

Proof:

Let X_0 *be an arbitrary non − zero time vector. A non − zero time vector* **Y** *is never orthogonal to* X_0 *because otherwise by* ***lemma 4,*** **Y** *would be a space vector so* ${}^tYGX_0 > 0$ *or* ${}^tYGX_0 < 0$.

We say that $Y \in \mathcal{C}_1$ *in the first case otherwise* $Y \in \mathcal{C}_2$. *The classes are opposite : if* $Y \in \mathcal{C}_1$ *then* $-Y \in \mathcal{C}_2$ *•The* ***lemm****a 7 shows that :*

$$\left({}^tX_0GY\right)\left({}^tYGZ\right)\left({}^tZGX_0\right) \geq \left({}^tX_0GX_0\right)\left({}^tYGY\right)\left({}^tZGZ\right) \geq 0,\ \forall Y,\ \forall Z \text{ time vectors}.$$

If **Y** *and* **Z** *belong*

− *to the same class* $\mathcal{C}_1$: ${}^tX_0GY \geq 0$ *et* ${}^tZGX_0 \geq 0 \Rightarrow {}^tYGZ \geq 0$,

− *to the same class* $\mathcal{C}_2$: ${}^tX_0GY \leq 0$ *et* ${}^tZGX_0 \leq 0 \Rightarrow {}^tYGZ \geq 0$

− *to different classes :* ${}^tX_0GY \leq 0$ *et* ${}^tZGX_0 \geq 0 \Rightarrow {}^tYGZ \leq 0$ *or*

${}^tX_0GY \geq 0$ *et* ${}^tZGX_0 \leq 0 \Rightarrow {}^tYGZ \leq 0$.

Conversely : If ${}^tYGZ \geq 0$

either ${}^tX_0GY \geq 0$ *and* ${}^tZGX_0 \geq 0 \Rightarrow$ **Y** *and* **Z** *belong to the same class* $\mathcal{C}_1$,

or ${}^tX_0GY \leq 0$ *and* ${}^tZGX_0 \leq 0 \Rightarrow$ **Y** *and* **Z** *belong to the same class* $\mathcal{C}_2$,

in the same way if ${}^tYGZ \leq 0 \Rightarrow$ **Y** *et* **Z** *belong to different classes* .

The last part of the lemma is a direct consequence of ***lemma 3****.*

Note : In an arbitrary way the elements of one of the **2** *classes are called vectors of future, the elements of the other vectors of past.*

<u>Lemme</u> *<u>9 :</u>*

Soient **X** *et* **Y** **2** *vecteurs de temps ou isotropes .*

Si **X** *et* **Y** *appartiennent à la même classe , leur somme est encore un vecteur de la même classe , et ils vérifient la contre − inégalité triangulaire si on note* $\|X\|_G = \sqrt{{}^tXX}$ *si* ${}^tXX \geq 0$ **:**

$$\|X+Y\|_G \geq \|X\|_G + \|Y\|_G.$$

Proof:

From the ***lemma 8*** ${}^tXGY \geq 0$ *and from the* ***lemm****a* ***6*** $\left|{}^tXGY\right| \geq \|X\|_G\|Y\|_G$ *and then* $XGY \geq \|X\|_G\|Y\|_G$, *therefore :*

$$\|X+Y\|_G^{\,2} = {}^t\!\left(X+Y\right)G\left(X+Y\right) = \|X\|_G^{\,2} + \|Y\|_G^{\,2} + 2\,{}^tXGY$$
$$\geq \|X\|_G^{\,2} + \|Y\|_G^{\,2} + + 2\,\|Y\|_G\|X\|_G = \left(\|X\|_G + \|X\|_G\right)^2.$$

If **V** *is a vector taken in the same class of* **X** *et* **Y** *, we have :*

${}^{t}XGV \geq 0$ *and* ${}^{t}YGV \geq 0 \Rightarrow {}^{t}(X+Y)GV \geq 0$ *and* $X+Y$ *belons to the class of* V *therefore* **to** *the class of* **X and Y**.

Note :

(1) *If* **X** *and* **Y** *are* **2** *vectors of class different the sum can be of any kind :*

If $X = {}^{t}(2, 1, 1, 1)$, ${}^{t}XGX = 1$ *and* $Y = {}^{t}(-3, 1, 1, 1)$, ${}^{t}YGY = 6$, *then* ${}^{t}XGY = -3$,

$X+Y = {}^{t}(-1, 2, 2, 2)$, ${}^{t}(X+Y)G(X+Y) = -11$ (*space vector*) **.**

If $X = {}^{t}(4, 1, 1, 1)$, ${}^{t}XGX = 13$ *and* $Y = {}^{t}(-1, 0, 0, 0)$, ${}^{t}YGY = 1$, *then* ${}^{t}XGY = -4$,

$X+Y = {}^{t}(3, 1, 1, 1)$, ${}^{t}(X+Y)G(X+Y) = 6$ (*time vector*).

If $X = {}^{t}(3, 1, 1, 1)$, ${}^{t}XGX = 6$ *and* $Y = {}^{t}\left(-2, \frac{\sqrt{3}}{3} - 1, \frac{\sqrt{3}}{3} - 1, \frac{\sqrt{3}}{3} - 1\right)$, ${}^{t}YGY = \frac{2\sqrt{3}}{3}$,

then ${}^{t}XGY = -3$, $X+Y = {}^{t}\left(1, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$,

${}^{t}(X+Y)G(X+Y) = 0$ (*isotropic vector*).

(2) *The triangular counter inequality gives a geometric explanation of the twin* $-$ *paradox* .

(6) Classification of Lorentz matrices.

We have seen that the set of Lorentz matrices $\mathcal{L}o$ of order **n** forms a subgroup of $GL_4(\mathbb{R})$, the group of invertible matrices. We can write a Lorentz matrix M in the most general writing :

$$M = exp(\alpha N)\begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix} \text{ where } \alpha \in \mathbb{R},\ N = \begin{bmatrix} 0 & {}^tX \\ X & O \end{bmatrix},\ X \in \mathbb{R}^3$$

$$\text{As } {}^tMGM = G \text{ avec } G = \begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbb{R}3} \end{bmatrix},\ det(M) = \pm 1.$$

But $det(exp(\alpha N)) = e^{Tr(\alpha N)} = e^0 = 1$ therefore

$$det(M) = det\left(\begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix}\right) = \varepsilon \cdot det(\Omega).$$

(2) Determination of ε

Let V_0 be a non $-$ zero time vector ${}^tV_0GV_0 > 0$ and M a Lorentz matrix. We have :

${}^t(MV_0)G(MV_0) = {}^tV_0\,({}^tMGM)\,V_0 = {}^tV_0GV_0 > 0$ so MV_0 is also a time vector.

Let $\eta = sign\left({}^tV_0G(MV_0)\right)$ si $\eta = 1$, V and (MV_0) belong to the same class otherwise $\eta = -1$, V and (MV_0) belong to different classes.

Now let V be another non $-$ zero time vector. As previously MV is also a time vector.

As ${}^t(MV)G(MV_0) = {}^tV\,({}^tMGM)\,V_0 = {}^tVGV_0$, we have the equivalence :

$Class(V) = Class(V_0) \Leftrightarrow Class(MV) = Class(MV_0)$.

So η depends only on M and is independent of V.

Let us calculate η for the vector $E_0 = {}^t(1, 0, 0, 0)$:

$$E_0ME_0 = [1, 0, 0, 0]\begin{bmatrix} \gamma & {}^t[\gamma\vec{\beta}] \\ [\gamma\vec{\beta}] & C \end{bmatrix}\begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \gamma\varepsilon \text{ therefore } \eta = \varepsilon.$$

If $\varepsilon = 1$ we will say that M is orthochronous otherwise if $\varepsilon = -1$, we will say that M is antichronous.

As $det(M) = \varepsilon \cdot det(\Omega)$ we have

$det(M) = +1 \Leftrightarrow \varepsilon = +1$ and $det(\Omega) = +1$ or $\varepsilon = -1$ and $det(\Omega) = -1$.

And if $det(M) = -1 \Leftrightarrow \varepsilon = -1$ et $det(\Omega) = +1$ or $\varepsilon = +1$ and $det(\Omega) = -1$.

Let be the following subsets of $\mathcal{L}o$:

$\mathcal{R}oo = \{M / \varepsilon = +1 \text{ et } det(\Omega) = +1\}$, the orthochronous rotation group,

$\mathcal{R}oo$ is a group called the restricted Lorentz groupsincefor **2** matrices of $\mathcal{R}oo$: M et M', we have :

$det(MM'^{-1}) = det(M)det^{-1}(M') = \varepsilon \cdot det(\Omega)\varepsilon' \cdot det(\Omega') = +1$.

$\mathcal{R}oa = \{M / \varepsilon = -1 \text{ et } det(\Omega) = +1\}$, the **antichronous rotations**,

$\mathcal{R}eo = \{M / \varepsilon = +1 \text{ et } det(\Omega) = -1\}$, the **orthochron**ous inversions,

$\mathcal{R}ea = \{M / \varepsilon = -1 \text{ et } det(\Omega) = -1\}$, the **antichronous** inversions .

We note $\mathcal{R}oo \cup \mathcal{R}oa = \{M / det(\Omega) = +1\}$ is a group, the **rotation group**,

and $\mathscr{R}oo \cup \mathscr{R}eo = \{ \boldsymbol{M} / \boldsymbol{\varepsilon} = +\boldsymbol{1} \}$, *the* ***orthochronous group,***
and $\mathscr{R}oo \cup \mathscr{R}ea = \{ \boldsymbol{M} / \boldsymbol{\varepsilon} = +\boldsymbol{1}$ and $\boldsymbol{det}(\boldsymbol{\Omega}) = +\boldsymbol{1}$ or $\boldsymbol{\varepsilon} = -\boldsymbol{1}$ and $\boldsymbol{det}(\boldsymbol{\Omega}) = -\boldsymbol{1} \}$, *the* ***pair group.***

We can also consider the set of matrices of Lorentz such as $\Omega = \boldsymbol{Id}_{\mathbb{R}^3}$, $\boldsymbol{\varepsilon} = +\boldsymbol{1}$ *and* $\vec{\beta} = \beta_x \vec{i}$:

$$\textit{Therefore } \boldsymbol{M} = \begin{bmatrix} \gamma & \gamma\beta_x & 0 & 0 \\ \gamma\beta_x & \gamma & 0 & 0 \\ \multicolumn{2}{c}{\mathbb{T}} & \multicolumn{2}{c}{Id_{\mathbb{R}^2}} \end{bmatrix} \textit{ with } \mathbb{T} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is easy to check that this set is a group of composition : the special group or boost :
We also note that :

$$\boldsymbol{MM'} = \begin{bmatrix} \gamma & \gamma\beta_x & 0 & 0 \\ \gamma\beta_x & \gamma & 0 & 0 \\ \multicolumn{2}{c}{\mathbb{T}} & \multicolumn{2}{c}{Id_{\mathbb{R}^2}} \end{bmatrix} \begin{bmatrix} \gamma' & \gamma'\beta'_x & 0 & 0 \\ \gamma'\beta'_x & \gamma' & 0 & 0 \\ \multicolumn{2}{c}{\mathbb{T}} & \multicolumn{2}{c}{Id_{\mathbb{R}^2}} \end{bmatrix} = \gamma\gamma' \begin{bmatrix} 1+\beta_x\beta'_x & \beta_x + \beta'_x & 0 & 0 \\ \beta_x + \beta'_x & 1+\beta_x\beta'_x & 0 & 0 \\ \multicolumn{2}{c}{\mathbb{T}} & \multicolumn{2}{c}{Id_{\mathbb{R}^2}} \end{bmatrix}.$$

We notice that this group is commutative .